Ternary semi-direct products in algebraically coherent categories

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Actions in semi-abelian categories

In a semi-abelian category $\mathcal{C}$ [1], there is a correspondence between *internal actions* [2] and *split short exact sequences*. Indeed, there is an adjunction

$$\mathbf{SSES}(\mathcal{C}) \underset{\Sigma}{\overset{(\mathrm{Ker},\mathrm{Cod})}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathcal{C}\times\mathcal{C}$$

$$(A \xrightarrow{j} X \underset{s}{\overset{p}{\rightleftarrows}} B) \longmapsto (A, B)$$

$$(B\flat A \xrightarrow{\kappa} B + A \underset{\iota}{\overset{[1,0]}{\rightleftarrows}} B) \longmapsto\!\dashv (A, B)$$

and the category $\mathbf{Act}(\mathcal{C})$ of actions of $\mathcal{C}$ is the category of algebras $\xi^B_A : B\flat A \to A$ for the induced monad.

The category $\mathbf{Act}_B(\mathcal{C})$ is the category of algebras of the restricted monad $A \mapsto B\flat A$, corresponding to the restricted adjunction $\Sigma_B \dashv \mathrm{Ker}_B$.

Semi-direct products

This adjunction is monadic, so that every object in $\mathbf{SSES}(\mathcal{C})$ is of the form

$$B \flat A \xrightarrow{\xi_A^B} A \qquad \Leftrightarrow \qquad A \xrightarrow{j_A} A \rtimes B \underset{s_B}{\overset{p_B}{\rightleftarrows}} B$$

and every morphism is of the form

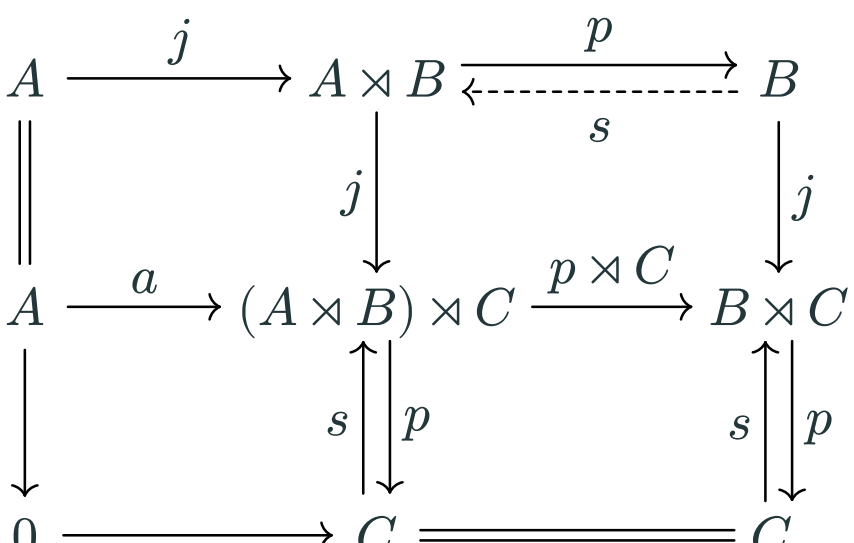
$$\begin{array}{ccc} B \flat A & \xrightarrow{\xi_A^B} & A \\ {\scriptstyle \beta \flat \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B' \flat A' & \xrightarrow[\xi_{A'}^{B'}]{} & A' \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{j_A} & A \rtimes B & \underset{s_B}{\overset{p_B}{\rightleftarrows}} & B \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \alpha \rtimes \beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ A' & \xrightarrow[j_{A'}]{} & A' \rtimes B' & \underset{s_{B'}}{\overset{p_{B'}}{\rightleftarrows}} & B' \end{array}$$

Ternary semi-direct products

Definition An object X in a semi-abelian category is a *ternary semi-direct product* [3], [4] of its subobjects A, B, C if

- $X = A \vee B \vee C$;
- A and $A \vee B$ are normal in X;
- $A \wedge B = 0 = (A \vee B) \wedge C$.

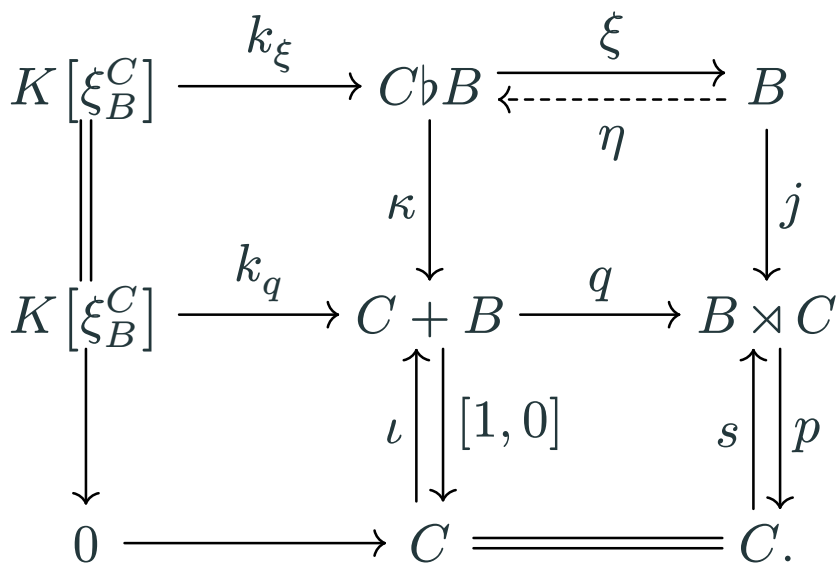
Every ternary semi-direct product induces a diagram



Ternary semi-direct products

Example Every binary coproduct $C + B$ can be seen as a ternary semi-direct product.

Indeed, we have a diagram

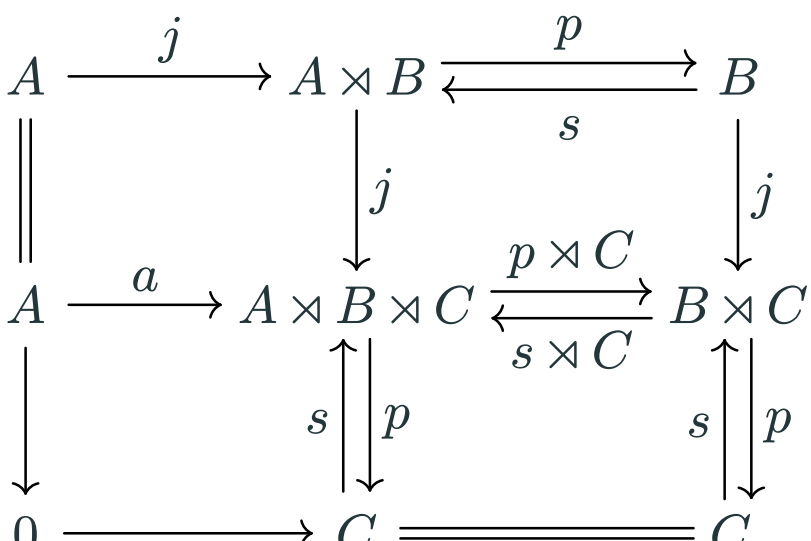


Ternary semi-direct products

Proposition (D.) If (B, C, ξ) is an action, then $\mathbf{Act}_{B \rtimes C}(\mathcal{C}) \cong \mathbf{Act}_{(B,\xi)}(\mathbf{Act}_C(\mathcal{C}))$. In particular, given actions $\xi_A^B, \xi_A^C, \xi_B^C$, there is a bijection between

1. actions of $B \rtimes C$ on A extending ξ_A^B and ξ_A^C
2. actions of (B, ξ_B^C) on (A, ξ_A^C) in $\mathbf{Act}_C(\mathcal{C})$ above (A, B, ξ_A^B)
3. actions of C on $A \rtimes B$ preserving the split short exact sequence.

This gives a ternary semi-direct product, with a diagram



The structure of ternary semi-direct products

Proposition (Carrasco, Cegarra) A ternary semi-direct product in $\mathbf{Grp}$ or $\mathbf{Lie}$ is determined by actions ξ_A^B, ξ_A^C, ξ_B^C and a function $C \times B \to A$, satisfying certain identities.

Proposition (D.) A ternary semi-direct product in a semi-abelian category $\mathcal{C}$ is uniquely determined by

- actions $\xi_A^{C+B} : (C+B) \flat A \to A$ and $\xi_B^C : C \flat B \to B$
- a $(C+B)$-equivariant morphism $\varphi : K\left[\xi_B^C\right] \to A$

such that $(\varphi \rtimes (C+B))^* \chi = [\ker(q_\xi), 1\rangle^* \xi_A^{C+B}$.

Change of base

The bifunctor $\flat : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induces a functor $\mathcal{C} \to \mathbf{Mnd}(\mathcal{C})$. Thus each $f : B \to C$ induces a functor $f^* : \mathbf{Act}_C(\mathcal{C}) \to \mathbf{Act}_B(\mathcal{C})$.

For a cospan $B \xrightarrow{b} X \xleftarrow{c} C$, we have two functors

$$\begin{array}{ccc} \mathbf{Act}_X(\mathcal{C}) & \xrightarrow{c^*} & \mathbf{Act}_C(\mathcal{C}) \\ {\scriptstyle b^*}\downarrow & & \downarrow \\ \mathbf{Act}_B(\mathcal{C}) & \longrightarrow & \mathcal{C} \end{array}$$

Question Given actions ξ_A^B and ξ_A^C, what can be said about a corresponding action ξ_A^X, for example if $X = B + C$ or $B \rtimes C$?

In general, neither existence nor uniqueness are guaranteed.

A generalized multiplication

There is a natural morphism $\gamma : C\flat(B\flat A) \longrightarrow (C+B)\flat A$.

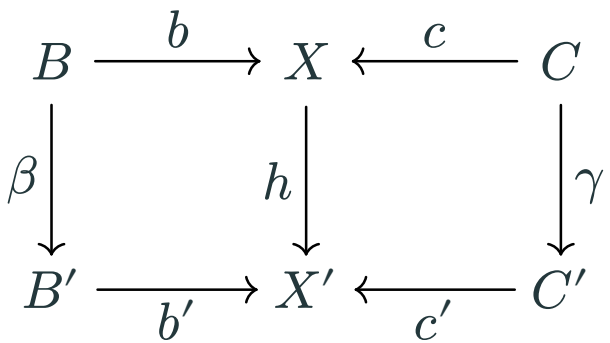
Given a cospan $B \xrightarrow{b} X \xleftarrow{c} C$, we can define a morphism $\nu_A^{c,b}$ as the composition of γ with $[c,b]\flat A$.

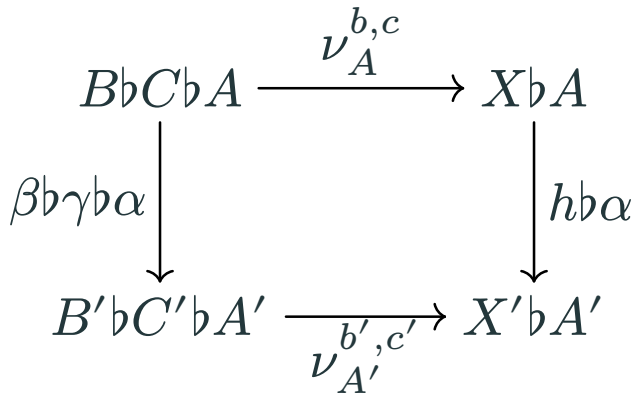
Alternatively, ν is the morphism which makes the following diagram commute :

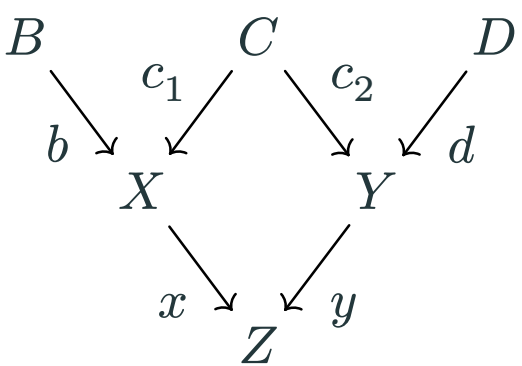
$$\begin{array}{ccccc}
C\flat B\flat A & \xrightarrow{\kappa} & C+B\flat A & \underset{\iota}{\overset{[1,0]}{\rightleftarrows}} & C \\
\downarrow{\scriptstyle \nu_A^{c,b}} & & \downarrow{\scriptstyle [\iota\circ c,\kappa\circ b\flat A]} & & \downarrow{\scriptstyle c} \\
X\flat A & \xrightarrow[\kappa]{} & X+A & \underset{\iota}{\overset{[1,0]}{\rightleftarrows}} & X
\end{array}$$

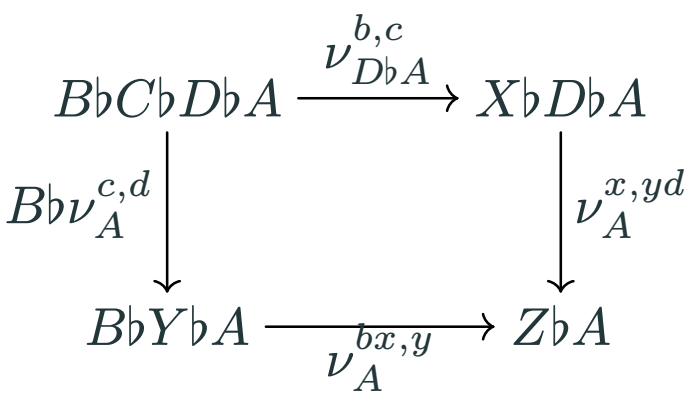
A generalized multiplication

This morphism $\nu_A^{b,c}$ is natural, and satisfies a form of associativity :









The LACC case

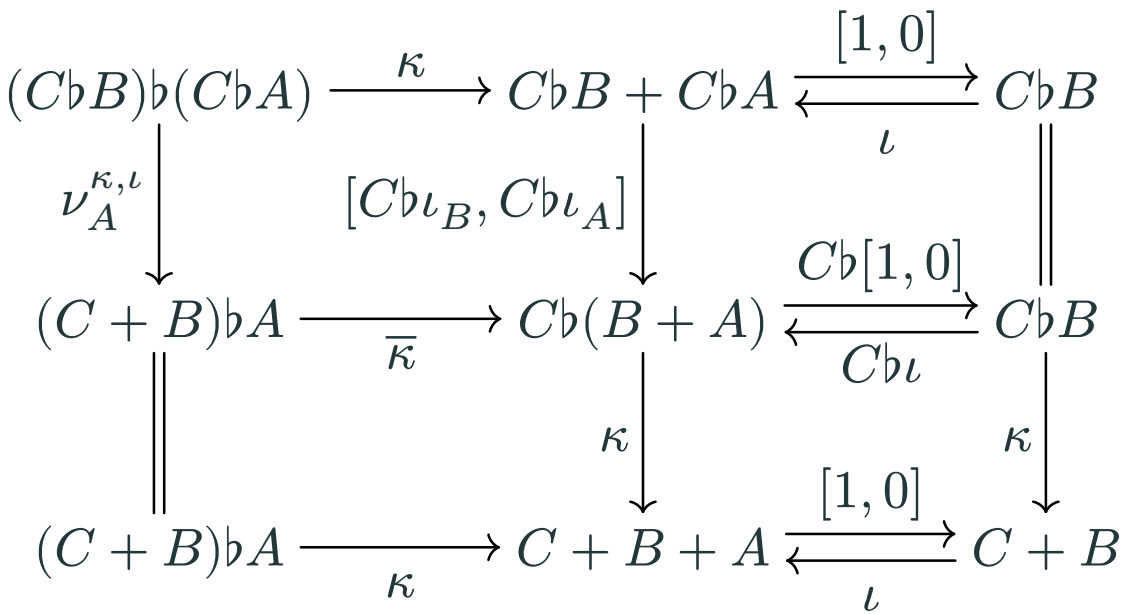
If $\mathcal{C}$ is *locally algebraically cartesian closed* [5], then $\mathbf{Act}_C(\mathcal{C}) \to \mathcal{C}$ creates coproducts, and it particular it also creates semi-direct products and actions.

Proposition (D.) The following conditions are equivalent :

1. each functor $C\flat_$ preserves finite coproducts;
2. $[C\flat\iota_B, C\flat\iota_A] : C\flat B + C\flat A \to C\flat(B + A)$ is an isomorphism;
3. $\nu_A^{\kappa,\iota} : (C\flat B)\flat(C\flat A) \to (C + B)\flat A$ is an isomorphism;
4. $\nu_A^{j,s} : B\flat C\flat A \to (B \rtimes C)\flat A$ is an isomorphism.

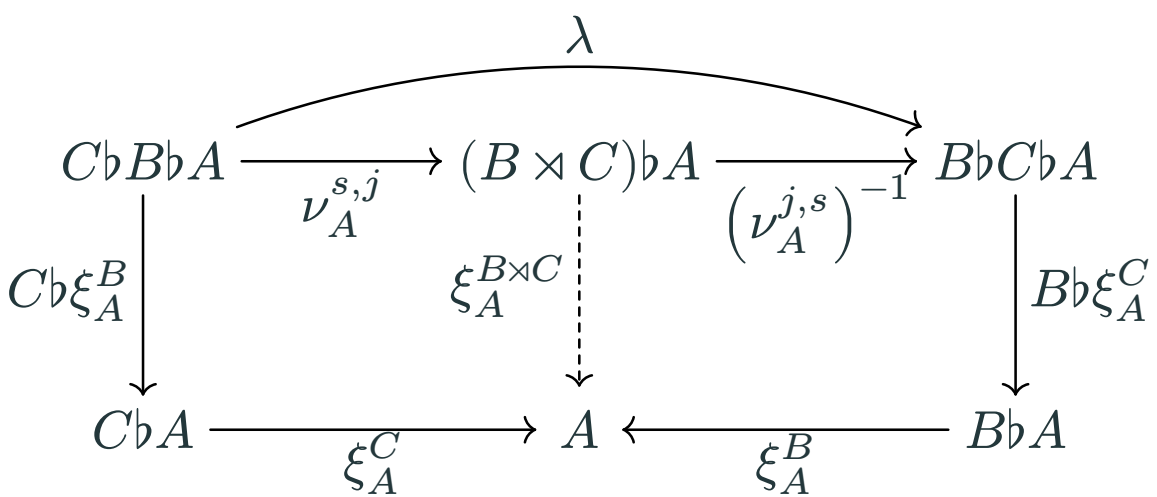
The LACC case

Proof sketch : We have a diagram



The LACC case

Proposition (D.) When the previous condition is satisfied, then $\lambda = \left(\nu^{j,s}\right)^{-1} \circ \nu^{s,j}$ is a distributive law between the monads of B and C-actions, and $\nu_A^{j,s}$ is an isomorphism of monads. In particular, there is a bijection between actions $\xi_A^{B \rtimes C}$ and pairs of actions ξ_A^B, ξ_A^C such that



Algebraically coherent categories

Definition A semi-abelian category $\mathcal{C}$ is *algebraically coherent* if the kernel functor $\mathrm{Ker}_C : \mathbf{SSES}_C(\mathcal{C}) \to \mathcal{C}$ is coherent for every C in $\mathcal{C}$ [6].

Proposition (Cigoli, Gray, Van der Linden) The following conditions are equivalent :

1. $\mathcal{C}$ is algebraically coherent;
2. each functor $C\flat_$ preserves jointly strongly epimorphic pairs;
3. $[C\flat\iota_B, C\flat\iota_A] : C\flat B + C\flat A \to C\flat(B + A)$ is a regular epimorphism.

For example, every category of interest is algebraically coherent. This property is also preserved by Birkhoff subcategories and category of functors.

Proposition (D.) The following conditions are equivalent :

1. $\mathcal{C}$ is algebraically coherent;
2. $\nu_A^{\kappa,\iota} : (C\flat B)\flat(C\flat A) \to (C + B)\flat A$ is a regular epimorphism
3. $\nu_A^{j,s} : B\flat C\flat A \to (B \rtimes C)\flat A$ is a regular epimorphism.

Algebraically coherent categories

Theorem (D.) Let $\mathcal{C}$ be an algebraically coherent semi-abelian category, $\left(A, B, \xi_A^B\right)$, $\left(A, C, \xi_A^C\right)$ and $\left(B, C, \xi_B^C\right)$ be actions, and $\xi : (B \rtimes C) \flat A \to A$ be a morphism. Then the following conditions are equivalent :

1. ξ is an action, and $j_B^* \xi = \xi_A^B$ and $s_C^* \xi = \xi_A^C$
2. the following diagram commutes :

$$\begin{array}{ccccc} C\flat B\flat A & \xrightarrow{\nu_A^{s,j}} & (B \rtimes C)\flat A & \xleftarrow{\nu_A^{j,s}} & B\flat C\flat A \\ \downarrow{\scriptstyle C\flat \xi_A^B} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle B\flat \xi_A^C} \\ C\flat A & \xrightarrow{\xi_A^C} & A & \xleftarrow{\xi_A^B} & B\flat A \end{array}$$

Moreover, there is at most one such ξ, and $\mathbf{Act}_{B\rtimes C}(\mathcal{C})$ is a full subcategory of $\mathbf{Act}_B(\mathcal{C}) \times_{\mathcal{C}} \mathbf{Act}_C(\mathcal{C})$.

Ternary semi-direct products

Theorem (D.) Let $\mathcal{C}$ be an algebraically coherent semi-abelian category, and A, B, C be objects of $\mathcal{C}$, together with

- actions $\xi_A^B, \xi_A^C, \xi_B^C$
- a B and C-equivariant morphism $\varphi : K = K\left[\xi_B^C\right] \to A$

If $\xi : (C + B) \flat A \to A$ is a morphism, then the following conditions are equivalent :

1. ξ is an action, extending $\varphi^* \chi_A, \xi_A^B, \xi_A^C$
2. the following diagram commutes :

$$\begin{array}{ccccccc}
C\flat B\flat A & \xrightarrow{\gamma} & (C+B)\flat A & \xleftarrow{\nu_A^{\kappa,\iota}} & (C\flat B)\flat(C\flat A) & \xleftarrow{\nu_{C\flat A}^{k,\eta}} & K\flat B\flat C\flat A \\
\downarrow{\scriptstyle C\flat\xi_A^B} & & \downarrow{\scriptstyle \xi} & & & & \downarrow{\scriptstyle K\flat B\flat\xi_A^C} \\
C\flat A & \xrightarrow{\xi_A^C} & A & \xleftarrow{\varphi^*\chi_A} & K\flat A & \xleftarrow{K\flat\xi_A^B} & K\flat B\flat A
\end{array}$$

There is at most one such ξ, and this dermines a ternary semi-direct product.

References

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