On the exactness of categories of internal groups

from the perspective of cosets and stabilizers

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Main idea

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1. **Premise**
   A finitely complete category $\mathcal{C}$ may not be exact while $\mathrm{Grp}(\mathcal{C})$ is (e.g. $\mathcal{C} = \mathrm{Comon}_k^{\mathrm{coc}}, \mathbf{Cat}$). Why is that?

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   Stabilizer subgroups seem to be the key to this puzzle. With enough effective equivalence relations, subgroups upgrade to stabilizers in $\mathrm{Grp}(\mathcal{C})$. If $\mathcal{C}$ has, and its product preserves, reflexive coequalizers, then equivalence relations in $\mathrm{Grp}(\mathcal{C})$ being effective is characterized by Bourn-normal subgroups being stabilizers.

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3. **Inspiration**
   The main influence for this work comes from the 2019 paper "A semi-abelian extension of a theorem by Takeuchi" by Gran, Sterck and Vercruysse [4].

Exactness

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A regular category is **exact** if its equivalence relations are kernel pairs [1].

Pullback regular categories have a stable orthogonal factorization system (**Regular epi**, **Mono**).

Stabilizers

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Definition 2 (Stabilizer)

A morphism $n \to h$ in $\mathcal{C}$ is a **stabilizer** subobject of h, if there is a morphism $f : h \to M$ in $\mathcal{C}^h$ so that $n \to h = f^{-1}(f(e)) =: \mathrm{Stab}_f$:

$$\begin{array}{ccc} n & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle fe} \\ h & \xrightarrow[f]{} & M \end{array}$$

Every stabilizer is a subgroup, and Stab defines a functor $h\backslash\mathcal{C}^h \to \mathrm{Grp}(\mathcal{C})/h$.

Basic properties of stabilizers

Stabilizers share the closure properties of regular monos.

Regular subgroups as stabilizers

If $g \xrightarrow{f} h \underset{\psi}{\overset{\phi}{\rightrightarrows}} k$ is an equalizer, then $f = \mathrm{Stab}_{e_k}$, where h acts on k by

$$x.y \coloneqq \phi(x) y \psi(x)^{-1}, \text{ for } x \in h, y \in k.$$

Stabilizers closed under pullbacks

$$\begin{array}{ccccc} \mathrm{Stab}_{ff'} & \longrightarrow & \mathrm{Stab}_f & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle fe_h} \\ g & \xrightarrow[f']{} & h & \xrightarrow[f]{} & M \end{array}$$

More basic properties of stabilizers

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Closure under intersection

If $f_i \colon h \to M_i$ is a morphism in $\mathcal{C}^h$, for $i = 1, 2$, then $\mathrm{Stab}_{f_1} \cap \mathrm{Stab}_{f_2} = \mathrm{Stab}_{(f_1, f_2)}$, where $(f_1, f_2) \colon h \to M_1 \times M_2$.

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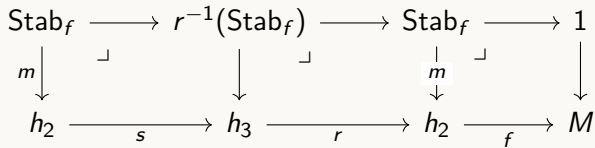
Closure under post-composition with sections

Consider a retraction $r \colon h_3 \to h_2$ with a splitting s in $\mathrm{Grp}(\mathcal{C})$ and a stabilizer $m = \mathrm{Stab}_f \colon h_1 \to h_2$. Now $sm = s \cap r^{-1}(\mathrm{Stab}_f)$:



Coset relations and subgroups

Coset relations

The **coset relation** of a subgroup $g \hookrightarrow h$ is the equivalence relation

$$(\pi_1, \mu|)\colon h \times g \rightrightarrows h, \qquad (x, y) \mapsto x,\ xy$$

in $\mathcal{C}^h$. This gives a bijection $\mathrm{Sub}_{\mathrm{Grp}(\mathcal{C})}(h) \cong \mathrm{EqRel}_{\mathcal{C}^h}(h)$, with inverse

$$(r_1, r_2\colon R \rightrightarrows h) \mapsto (r_2 \ker(r_1)\colon k_1 \hookrightarrow h) = (0, \mathrm{id}_h)^{-1}(R).$$

Restrictions

- Stabilizers $\leftrightarrow$ effective coset relations (kernel pairs in $\mathcal{C}^h$), by the shear isomorphism $h \times \mathrm{Stab}_f \cong h \times_M h$, $(x, y) \mapsto (x, xy)$.
- **Bourn-normal** (conjugation-closed) subgroups $\leftrightarrow$ equivalence relations in $\mathrm{Grp}(\mathcal{C})$ [2, Prop. 3.2.12].

Coset quotients and stabilizers

Theorem 3

Suppose $\mathcal{C}$ has, and $h \times (-)$ preserves, reflexive coequalizers. Then $\mathcal{C}^h \to \mathcal{C}$ strictly creates reflexive coequalizers, and ***coset quotients*** *define a left adjoint to* Stab*:*

$$Q\colon \mathrm{Grp}(\mathcal{C})/h \to h\backslash\mathcal{C}^h, \qquad (g \to h) \mapsto \mathrm{Coeq}(h \times g \rightrightarrows h)\colon h \twoheadrightarrow h/g.$$

Corollary 4

With the above assumptions, $Q \dashv$ Stab restricts to the latter isomorphism of posets

$$\{\textit{effective coset relations on } h\} \cong \{\textit{stabilizers of } h\} \cong \mathrm{RegQuot}_{\mathcal{C}^h}(h).$$

Exactness of $\mathrm{Grp}(\mathcal{C})$

Assume $\mathcal{C}$ has, and its product preserves, reflexive coequalizers. Then $\mathrm{Grp}(\mathcal{C}) \to \mathcal{C}$ strictly creates limits and reflexive coequalizers, since each $x \mapsto x^n \colon \mathcal{C} \to \mathcal{C}$ preserves reflexive coequalizers.

Theorem 5

Every regular epi in $\mathrm{Grp}(\mathcal{C})$ *is stably stabilizer-extremal if the following condition holds:*

- *The composite* $g \twoheadrightarrow h \twoheadrightarrow M$ *is a regular epi in* $\mathcal{C}$ *for regular epis* $g \twoheadrightarrow h$ *in* $\mathrm{Grp}(\mathcal{C})$ *and* $h \twoheadrightarrow M$ *in* $\mathcal{C}^h$.

Proof of main theorem

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Let $f : g \twoheadrightarrow h$ be a regular epi in $\mathrm{Grp}(\mathcal{C})$ with a pullback f'. We may assume the pullback is along a section s; otherwise replace f by $\mathrm{id} \times f$. Assume $f' = me$, where m is a stabilizer. Consider the diagram on the right.

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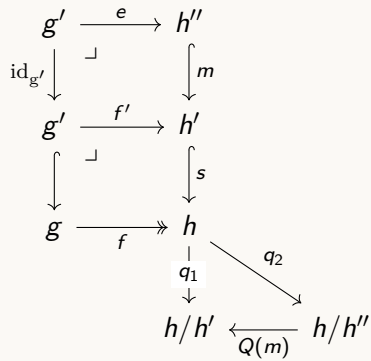
By assumption $q_1 f$ and $q_2 f$ are coset quotients with the same stabilizer g'. Thus $Q(m)$ is an isomorphism and so m is an isomorphism.



□

Corollaries

Standing assumption: $\mathcal{C}$ is finitely complete and has reflexive coequalizers, preserved by the product componentwise.

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Corollary 6 (Surjectivity of Newman's correspondence implies stability)

If extremal coset quotients $h \twoheadrightarrow M$ are regular, then every regular epi in $\mathsf{Grp}(\mathcal{C})$ *is stably stabilizer-extremal, especially stably epi. The hypothesis holds when regular epis in $\mathcal{C}$ are closed under composition.*

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Corollary 7 (Newman's correspondence implies exactness of $\mathsf{Grp}(\mathcal{C})$)

If extremal coset quotients are regular and coset relations are effective, then $\mathsf{Grp}(\mathcal{C})$ *is exact. Furthermore, if regular epis in $\mathcal{C}$ are closed under composition and coset relations are effective, then* $\mathsf{Grp}(\mathcal{C})$ *is exact.*

Regularity of stabilizers

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Theorem 8

Let $h \xrightarrow{f} M \xhookrightarrow{j} A$ be morphisms in $\mathcal{C}^h$, where h is an internal group, A is an h-group, and j is monic. Then the stabilizer $\mathrm{Stab}_f \hookrightarrow h$ of f is a regular mono.

Proof.

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Proof.

Let $A \overset{i}{\hookrightarrow} A \rtimes h \rightleftarrows h$ be the split extension in $\mathrm{Grp}(\mathcal{C})$, with section s. Write for $x\colon h$ and $a\colon A$

- $x.a = s(x)as(x)^{-1}$
- $m_0 \coloneqq fe_h$, $a_0 \coloneqq j(m_0)$
- $\phi_{a_0} \coloneqq a_0(-)a_0^{-1}$ on $A \rtimes h$

$$
\begin{aligned}
x \in \mathrm{Stab}_f &\iff x.m_0 = m_0 \\
&\iff x.a_0 = a_0 \ (j \text{ equivariant monic}) \\
&\iff s(x)a_0s(x)^{-1} = a_0 \\
&\iff \phi_{a_0}(s(x)) = s(x) \\
&\iff x \in \mathrm{Eq}(s, \phi_{a_0}s).
\end{aligned}
$$

□

Regularity of stabilizers

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Corollary 9

If h is exponentiable ($h \times (-)$ has a right adjoint $[h, -]$) in $\mathcal{C}$ and $f : h \to M$ is a morphism in $\mathcal{C}^h$ with a monic $j' : M \to g$ in $\mathcal{C}$ to an internal group g, then the stabilizer of f is regular.

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Proof.

Since there is a monic $j' : M \to g$ in $\mathcal{C}$, we obtain an h-equivariant monic $j : M \to [h, g]$. Since $[h, g]$ is an h-group [5, Theorem 5.3], the stabilizer of f is regular. □

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Corollary 10

If $\mathcal{C}$ is cartesian closed with each object embedding into an internal group, then stabilizer subgroups are regular. If, moreover, each coset relation is effective, then every subgroup is regular.

Cocommutative Hopf algebras

Corollary 11

Every monomorphism of cocommutative Hopf algebras is regular.

Proof.

$\mathrm{Comon}_k^{\mathrm{coc}}$ is cartesian closed, and every cocommutative coalgebra embeds into its free Hopf algebra, which is cocommutative [7]. Coset relations are effective by Newman's correspondence [6], so Corollary 10 applies. □

Stabilizers organize both questions

Exactness

With $\mathcal{C}$ finitely compolete, with coequalizer of reflexive pairs that the product preserves, Newman's correspondence separates into effectivity of coset relations and regularity of coset quotients (epis). Its bijectivity then implies the exactness of cocommutative Hopf algebras [4].

Regular monomorphisms

Regular monos are stabilizers. If $\mathcal{C}$ is cartesian closed and has group embeddings, stabilizers are regular monos.

Thank you!

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