The center of an associative crossed module

Emzar Khmaladze Manuel Ladra David Quiroga Brais Ramos Pérez

Universidad de Santiago de Compostela

13/11/2026



FACULTADE DE MATEMÁTICAS

PID2024-155502NB-I00, MCIN/AEI/10.13039/501100011033 and from FEDER, UE,
Xunta de Galicia through the Competitive Reference Groups (GRC), ED431C

Contents

1. Associative crossed modules
2. Center of an associative crossed module
3. Huq center of an associative crossed module
4. Drinfeld-type interpretation of the center

Associative crossed modules

Definition (Associative action of A on M)

Bilinear maps $(a, m) \mapsto {}^{a}m$ and $(m, a) \mapsto m^{a}$ such that

$$
\begin{aligned}
{}^{ab}m &= {}^{a}({}^{b}m) & \qquad m^{ab} &= (m^{a})^{b} \\
{}^{a}(mn) &= ({}^{a}m)n & \qquad (mn)^{a} &= m(n^{a}) \\
({}^{a}m)^{b} &= {}^{a}(m^{b}) & \qquad (m^{a})n &= m({}^{a}n)
\end{aligned}
$$

Associative crossed modules

Definition (Associative action of A on M)

Bilinear maps $(a, m) \mapsto {}^{a}m$ and $(m, a) \mapsto m^{a}$ such that

$$
\begin{aligned}
{}^{ab}m &= {}^{a}({}^{b}m) & m^{ab} &= (m^{a})^{b} \\
{}^{a}(mn) &= ({}^{a}m)n & (mn)^{a} &= m(n^{a}) \\
({}^{a}m)^{b} &= {}^{a}(m^{b}) & (m^{a})n &= m({}^{a}n)
\end{aligned}
$$

Definition (Crossed module $(\mathsf{M}, \mathsf{A}, \mu)$)

An associative morphism $\mu\colon \mathsf{M} \to \mathsf{A}$ together with an action of A on M such that

$$
\begin{aligned}
&\mu({}^{a}m) = a\mu(m), \quad \mu(m^{a}) = \mu(m)a && (\mu \text{ A-equivariant}) \\
&{}^{\mu(m)}n = mn = m^{\mu(n)} && \text{(Peiffer identity)}
\end{aligned}
$$

Braidings

Commutators: $[a,b] = ab - ba, \quad [a,m] = {}^{a}m - m^{a}.$

Definition (Braiding)

A bilinear map $\{-,-\}\colon \mathsf{A} \times \mathsf{A} \to \mathsf{M}$ on $(\mathsf{M}, \mathsf{A}, \mu)$ such that

$$\mu\{a,b\} = [a,b]$$
$$\{\mu(m), a\} = -[a,m] \qquad \{a, \mu(m)\} = [a,m]$$
$$\{ab, c\} = {}^{a}\{b,c\} + \{a,c\}^{b} \qquad \{a, bc\} = {}^{b}\{a,c\} + \{a,b\}^{c}$$

A crossed module with a braiding is a braided associative crossed module.

1 Associative crossed modules

2 Center of an associative crossed module

3 Huq center of an associative crossed module

4 Drinfeld-type interpretation of the center

Center of an associative crossed module

Definition

$\mathbf{Z}(\mathsf{M},\mathsf{A},\mu) \coloneqq (\mathsf{M},\mathscr{T}(\mathsf{M},\mathsf{A},\mu),\partial)$ is the center of $(\mathsf{M},\mathsf{A},\mu)$.

Building the center I: bimultiplications

Definition

Let A act on M. A linear map $\lambda\colon A \to M$ is a *left multiplication* and $\rho\colon A \to M$ a *right multiplication* if

$$\lambda(ab) = \lambda(a)^b, \qquad \rho(ab) = {}^a\rho(b).$$

A *bimultiplication* is a pair (λ, ρ) of them with

$${}^a\lambda(b) = \rho(a)^b.$$

Building the center I: bimultiplications

Definition

Let A act on M. A linear map $\lambda\colon A \to M$ is a *left multiplication* and $\rho\colon A \to M$ a *right multiplication* if

$$\lambda(ab) = \lambda(a)^b, \qquad \rho(ab) = {}^a\rho(b).$$

A *bimultiplication* is a pair (λ, ρ) of them with

$${}^a\lambda(b) = \rho(a)^b.$$

Example

For $m \in M$ let $\mathscr{L}_m(a) := m^a$ and $\mathscr{R}_m(a) := {}^a m$, then $(\mathscr{L}_m, \mathscr{R}_m) \in \mathscr{B}\mathrm{Mul}(A, M)$

Building the center II: the vector space $\mathscr{T}(\mathsf{M}, \mathsf{A}, \mu)$ and ∂

Definition

$\mathscr{T}(\mathsf{M}, \mathsf{A}, \mu) \coloneqq \{(\lambda, \rho, a) \mid a \in \mathsf{A},\ (\lambda, \rho) \in \mathscr{B}\mathsf{Mul}(\mathsf{A}, \mathsf{M})\}$ such that, for all $b \in \mathsf{A}$, $m \in \mathsf{M}$,

$$\mu\lambda(b) = ab \qquad \lambda\mu(m) = {}^{a}m$$
$$\mu\rho(b) = ba \qquad \rho\mu(m) = m^{a}$$

Building the center II: the vector space $\mathscr{T}(\mathsf{M},\mathsf{A},\mu)$ and ∂

Definition

$\mathscr{T}(\mathsf{M},\mathsf{A},\mu) \coloneqq \{(\lambda,\rho,a) \mid a \in \mathsf{A},\ (\lambda,\rho) \in \mathscr{B}\mathsf{Mul}(\mathsf{A},\mathsf{M})\}$ such that, for all $b \in \mathsf{A}$, $m \in \mathsf{M}$,

$$\mu\lambda(b) = ab \qquad \lambda\mu(m) = {}^{a}m$$
$$\mu\rho(b) = ba \qquad \rho\mu(m) = m^{a}$$

Lemma

For every $m \in \mathsf{M}$, $\partial(m) \coloneqq (\mathscr{L}_m, \mathscr{R}_m, \mu(m)) \in \mathscr{T}(\mathsf{M},\mathsf{A},\mu)$. *So* $\partial\colon \mathsf{M} \to \mathscr{T}(\mathsf{M},\mathsf{A},\mu)$ *is linear.*

Lemma

For $(\lambda,\rho,a),(\lambda',\rho',b) \in \mathscr{T}(\mathsf{M},\mathsf{A},\mu)$*:*

$$\partial\lambda(b) = (\mathscr{L}_{\lambda(b)}, \mathscr{R}_{\lambda(b)}, ab) = (\mathscr{L}_{\rho'(a)}, \mathscr{R}_{\rho'(a)}, ab) = \partial\rho'(a).$$

$\mathscr{T}(\mathsf{M},\mathsf{A},\mu)$ *is an associative algebra with* $(\lambda,\rho,a)\cdot(\lambda',\rho',b) \coloneqq \partial\lambda(b) = \partial\rho'(a)$.

The center: action and braiding

Theorem

Let $(\mathsf{M}, \mathsf{A}, \mu)$ *be an associative crossed module. Then* $\partial\colon \mathsf{M} \to \mathscr{T}(\mathsf{M}, \mathsf{A}, \mu)$*, with*

$$\begin{aligned} \textit{action:} \quad & {}^{(\lambda,\rho,a)}m \coloneqq {}^{a}m, \qquad m^{(\lambda,\rho,a)} \coloneqq m^{a}, \\ \textit{braiding:} \quad & \{(\lambda,\rho,a),(\lambda',\rho',b)\} \coloneqq \lambda(b) - \lambda'(a), \end{aligned}$$

is a braided associative crossed module.

Definition

$\mathbf{Z}(\mathsf{M}, \mathsf{A}, \mu) \coloneqq (\mathsf{M}, \mathscr{T}(\mathsf{M}, \mathsf{A}, \mu), \partial)$ is the center of $(\mathsf{M}, \mathsf{A}, \mu)$.

1 Associative crossed modules

2 Center of an associative crossed module

3 Huq center of an associative crossed module

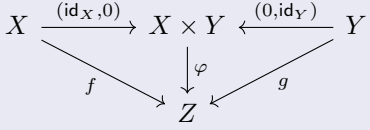
4 Drinfeld-type interpretation of the center

Huq center definition

Let $\mathscr{C}$ be pointed with finite products, and $f\colon X \to Z$, $g\colon Y \to Z$.

Definition

f and g Huq-commute if there is $\varphi\colon X \times Y \to Z$ with $\varphi(\mathrm{id}_X, 0) = f$ and $\varphi(0, \mathrm{id}_Y) = g$.

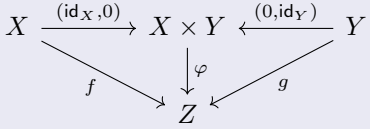


Huq center definition

Let $\mathscr{C}$ be pointed with finite products, and $f\colon X \to Z$, $g\colon Y \to Z$.

Definition

f and g Huq-commute if there is $\varphi\colon X \times Y \to Z$ with $\varphi(\mathrm{id}_X, 0) = f$ and $\varphi(0, \mathrm{id}_Y) = g$.



Definition

The Huq center $\mathcal{Z}^{\mathrm{Huq}}(X) \to X$ is the largest subobject that Huq-commutes with id_X (when it exists).

Huq center of an associative crossed module

Associative crossed modules form a semi-abelian category, in particular pointed with finite products.

Theorem

The Huq center of $(\mathsf{M}, \mathsf{A}, \mu)$ *is the abelian crossed module*

$$\mathcal{Z}^{\mathrm{Huq}}(\mathsf{M}, \mathsf{A}, \mu) = \big(\mathsf{Ann}_{\mathsf{M}}(\mathsf{A}),\, \mathcal{Z}_{\mathsf{M}}(\mathsf{A}),\, \mu|_{\mathsf{Ann}_{\mathsf{M}}(\mathsf{A})}\big).$$

Definition

For an action of A on M (with $\mathcal{Z}(\mathsf{A}) = \{a \mid ab = 0 = ba\ \forall b\}$):

$$\begin{aligned}
\mathsf{Ann}_{\mathsf{M}}(\mathsf{A}) &= \{m \in \mathsf{M} \mid \mathscr{L}_m = 0 = \mathscr{R}_m\} = \{m \mid m^a = 0 = {}^a m\ \forall a\},\\
\mathcal{Z}_{\mathsf{M}}(\mathsf{A}) &= \mathcal{Z}(\mathsf{A}) \cap \mathsf{Ann}_{\mathsf{A}}(\mathsf{M}),\\
\mathsf{Ann}_{\mathsf{A}}(\mathsf{M}) &= \{a \in \mathsf{A} \mid {}^a m = 0 = m^a\ \forall m\}.
\end{aligned}$$

Huq center versus the center $\mathbf{Z}$

Proposition

$(\mathrm{inc}, \zeta)\colon \mathcal{Z}^{\mathrm{Huq}}(\mathsf{M}, \mathsf{A}, \mu) \to \mathbf{Z}(\mathsf{M}, \mathsf{A}, \mu)$ *is an injective morphism of crossed modules.*

Define $\zeta\colon \mathcal{Z}_{\mathsf{M}}(\mathsf{A}) \to \mathscr{T}(\mathsf{M}, \mathsf{A}, \mu)$, $\zeta(a) = (0, 0, a)$. It is a well-defined injection.

$$\begin{array}{ccc} \mathsf{Ann}_{\mathsf{M}}(\mathsf{A}) & \xrightarrow{\mu|_{\mathsf{Ann}_{\mathsf{M}}(\mathsf{A})}} & \mathcal{Z}_{\mathsf{M}}(\mathsf{A}) \\ {\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle \zeta} \\ \mathsf{M} & \xrightarrow[\partial]{} & \mathscr{T}(\mathsf{M}, \mathsf{A}, \mu) \end{array}$$

1. Associative crossed modules
2. Center of an associative crossed module
3. Huq center of an associative crossed module
4. Drinfeld-type interpretation of the center

Drinfeld-type interpretation I: the category $\mathbf{Cat}(\mathrm{M}, \mathrm{A}, \mu)$

Category of associative crossed modules $\simeq$ category of internal categories in associative algebras (category of associative 2-algebras)

Definition

$\mathbf{Cat}(\mathrm{M}, \mathrm{A}, \mu)$

- Objects are the elements of A.
- Morphisms $a \xrightarrow{m} b$ with $\mu(m) + a = b$.

Drinfeld-type interpretation I: the category $\mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu)$

Category of associative crossed modules $\simeq$ category of internal categories in associative algebras (category of associative 2-algebras)

Definition

$\mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu)$

- Objects are the elements of A.
- Morphisms $a \xrightarrow{m} b$ with $\mu(m) + a = b$.

The product of A induces a bifunctor $\cdot \colon \mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu) \times \mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu) \to \mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu)$,

$$\left(a \xrightarrow{m} b\right) \cdot \left(a' \xrightarrow{m'} b'\right) = \left(aa' \xrightarrow{mm' + {}^{a}m' + m^{a'}} bb'\right),$$

in particular $\mathsf{id}_c \cdot m = {}^{c}m$ and $m \cdot \mathsf{id}_c = m^{c}$. This product is associative.

Drinfeld-type interpretation II: the center of $\mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu)$

Let $\mathbf{0}$ be the zero endofunctor. For $(\lambda, \rho, a) \in \mathscr{T}(\mathsf{M}, \mathsf{A}, \mu)$ and $b \in \mathsf{A}$ put

$$\sigma_b \colon 0 \xrightarrow{\lambda(b)} ab, \qquad \tau_b \colon 0 \xrightarrow{\rho(b)} ba$$

Drinfeld-type interpretation II: the center of $\mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu)$

Let $\mathbf{0}$ be the zero endofunctor. For $(\lambda, \rho, a) \in \mathscr{T}(\mathsf{M}, \mathsf{A}, \mu)$ and $b \in \mathsf{A}$ put

$$\sigma_b \colon 0 \xrightarrow{\lambda(b)} ab, \qquad \tau_b \colon 0 \xrightarrow{\rho(b)} ba$$

We can define natural transformations: $\sigma \colon \mathbf{0} \Rightarrow a \cdot (-)$, $\tau \colon \mathbf{0} \Rightarrow (-) \cdot a$. Moreover

$$\begin{aligned} \lambda \in \mathscr{L}\mathsf{Mul}(\mathsf{A}, \mathsf{M}) &\iff \sigma_{bc} = \sigma_b \cdot \mathsf{id}_c \\ \rho \in \mathscr{R}\mathsf{Mul}(\mathsf{A}, \mathsf{M}) &\iff \tau_{bc} = \mathsf{id}_b \cdot \tau_c \\ {}^a\lambda(b) = \rho(a)^b &\iff \mathsf{id}_b \cdot \sigma_c = \tau_b \cdot \mathsf{id}_c \end{aligned}$$

Drinfeld-type interpretation III: the main identification

Definition

$\mathbf{Z}(\mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu))$:

- Objects (σ, τ, a) with σ, τ as above.
- A morphism $(\sigma, \tau, a) \to (\sigma', \tau', b)$ is $a \xrightarrow{m} b$ in $\mathbf{Cat}(\mathsf{M}, \mathsf{A}, \mu)$ such that, for all $c \in \mathsf{A}$, the next diagram commute;

$$\begin{array}{ccc} 0 & \xrightarrow{\sigma_c} & ac \\ & \searrow^{\sigma'_c} & \downarrow{\scriptstyle m \cdot \mathsf{id}_c} \\ & & bc \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{\tau_c} & ca \\ & \searrow^{\tau'_c} & \downarrow{\scriptstyle \mathsf{id}_c \cdot m} \\ & & cb \end{array}$$

Drinfeld-type interpretation III: the main identification

Definition

$\mathbf{Z}(\mathbf{Cat}(\mathsf{M},\mathsf{A},\mu))$:

- Objects (σ,τ,a) with σ,τ as above.
- A morphism $(\sigma,\tau,a)\to(\sigma',\tau',b)$ is $a\xrightarrow{m}b$ in $\mathbf{Cat}(\mathsf{M},\mathsf{A},\mu)$ such that, for all $c\in\mathsf{A}$, the next diagram commute;

$$\begin{array}{ccc} 0 & \xrightarrow{\sigma_c} & ac \\ & \searrow^{\sigma'_c} & \downarrow{\scriptstyle m\cdot\mathsf{id}_c} \\ & & bc \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{\tau_c} & ca \\ & \searrow^{\tau'_c} & \downarrow{\scriptstyle \mathsf{id}_c\cdot m} \\ & & cb \end{array}$$

Theorem

There is a canonical isomorphism of categories

$$\mathbf{Z}\big(\mathbf{Cat}(\mathsf{M},\mathsf{A},\mu)\big)\ \cong\ \mathbf{Cat}\big(\mathbf{Z}(\mathsf{M},\mathsf{A},\mu)\big)=\mathbf{Cat}\big(\mathsf{M},\mathscr{T}(\mathsf{M},\mathsf{A},\mu),\partial\big).$$

Bibliography


S. Mac Lane, Extensions and obstructions for rings, *Illinois J. Math.* **2** (1958), 316–345


A. Fernández-Fariña and M. Ladra, Braiding for categorical algebras and crossed modules of algebras I: Associative and Lie algebras, *J. Algebra Appl.* **19** (9) (2020), 2050176, 30 pp.


M. Pirashvili, On the centre of crossed modules of Lie algebras, *J. Lie Theory* **35** (2) (2025), 359–375.