"2-nil or not 2-nil, that is the question": a characterisation of two-step nilpotent varieties

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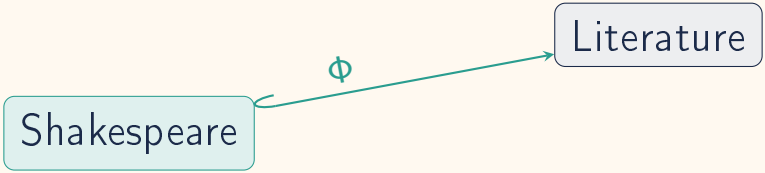
This is joint work with Manfred Hartl and Tim Van der Linden

XVI PORTUGUESE CATEGORY SEMINAR

12 November 2026, $(\mathrm{imbra})^{\mathrm{op}}$

$\mathcal{V} \simeq \mathrm{Mod}_{\mathcal{R}}$

Context: what if Shakespeare had been a mathematician?



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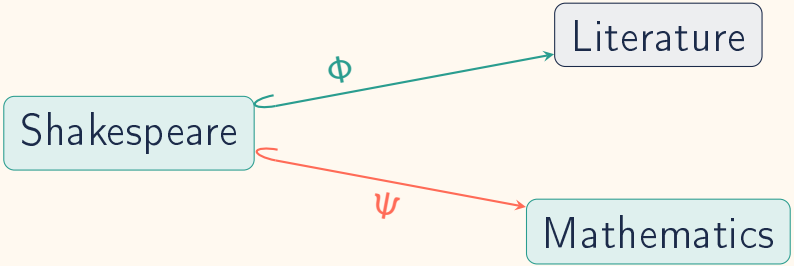

- The category Shakespeare is embedded in the category Literature.
- **Did you know?** It can also be embedded in the category Mathematics!

Context: what does Ψ do on objects?



Much Ado About Nothing $\longmapsto$ Much Ado About the Zero Object

Context: what does Ψ do on objects?



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That is the question...

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The aim of my talk is to answer this question **in the setting of varieties in the sense of universal algebra**.

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Characterisation of two-step nilpotent varieties

Thm. Let $\mathcal{V}$ be a variety of algebras. The following are equivalent:

(i) $\mathcal{V}$ is **two-step nilpotent**;

(ii) there exists a **square ring** $\mathcal{R}$ such that $\mathcal{V} \simeq \mathrm{Mod}_{\mathcal{R}}$.

[Geenis, Hartl, Van der Linden, 2026]

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What is a two-step nilpotent variety? **What is a module over a square ring?**

What is between abelian and semi-abelian?

Two-step nilpotency

Two ends of the spectrum

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$\mathcal{V}$ is semi-abelian $\iff$ it has

- a unique constant 0,
- n binary operations $\alpha_1, \ldots, \alpha_n$,
- an $(n+1)$-ary operation θ,

such that $\alpha_i(x, x) = 0$ and
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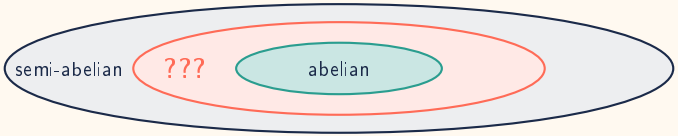

The binary commutator (Higgins)

Let X_1, X_2 be **subobjects** of X, with inclusions $x_i : X_i \rightarrowtail X$.

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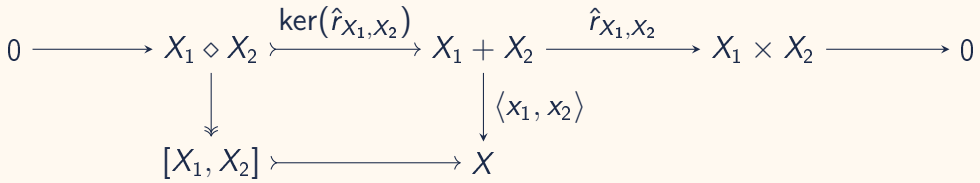

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Abelian varieties

$\mathcal{V}$ is abelian $\iff$ all binary commutators vanish: $[X, X] = 0$ for all X.

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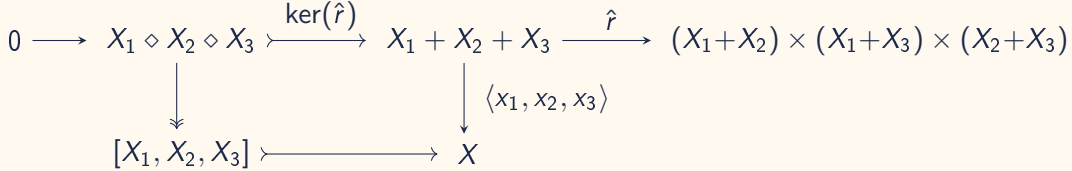

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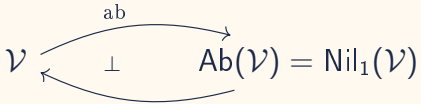
From semi-abelian to abelian, in two steps



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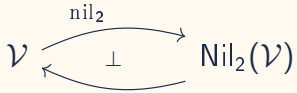
$\mathrm{ab}(X) = X/[X, X]$

semi-abelian $\to$ abelian



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$\mathrm{nil}_2(X) = X/[X, X, X]$

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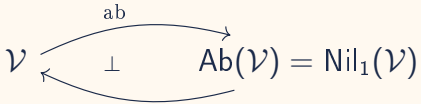

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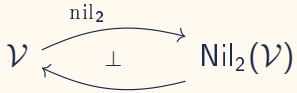

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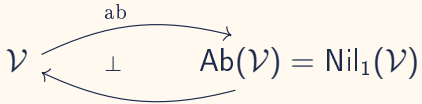

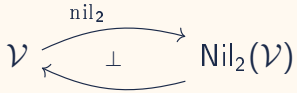

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Remark

1. We can define, similarly, n-ary commutators and $\mathsf{Nil}_n(\mathcal{V})$.
2. This construction also holds in semi-abelian **categories**.

Square rings and modules over them

The quadratic brother of rings

Where do square rings come from?

Origin: metastable homotopy theory, Baues–Hartl–Pirashvili (1997)

For $\Sigma\mathbb{RP}^2$, the suspended real projective plane, $\mathrm{End}(\Sigma\mathbb{RP}^2)$ gives a **square ring**, and the **free modules over it** form the homotopy category of Moore spaces $M(W, 2)$, where W is a $\mathbb{Z}/2\mathbb{Z}$-vector space.

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Two-step nilpotent world: quadratic

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- *Intuition:* degree ≤ 1.

Two-step nilpotent world: quadratic

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- *Intuition:* degree ≤ 2: a linear and a bilinear part, which interact.

A square ring, piece by piece

- $\mathcal{R}_{\mathrm{e}}$: a **(left) nearring** characterising the **unary** part.



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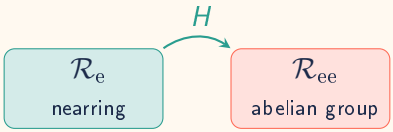

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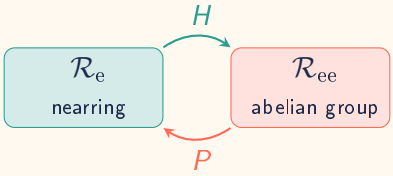

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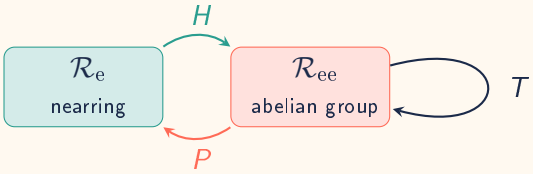

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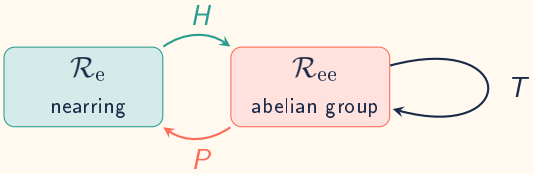

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These come with an action of $\mathcal{R}_{\mathrm{e}}$ on $\mathcal{R}_{\mathrm{ee}}$, and a few axioms making everything compatible.

Examples of square rings

Rings

Any ring R can be seen as the square ring $(R \xrightarrow{0} 0 \xrightarrow{0} R)$, where $\mathcal{R}_{\mathrm{ee}} = 0$ and $H = P = T = 0$.

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The nil square ring $\mathbb{Z}_{\mathrm{nil}}$

$$\mathbb{Z}_{\mathrm{nil}} = \left(\mathbb{Z} \xrightarrow{H} \mathbb{Z} \xrightarrow{P} \mathbb{Z}\right), \text{ with } H(r) = \frac{r(r-1)}{2} \text{ and } P = 0.$$

$$T = -1 \text{ and } (r, s) \cdot x \cdot t = rsxt.$$

Morphisms of square rings

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So, what is a module over such an object?

A module over a square ring, piece by piece

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The category of modules

$\mathrm{Mod}_{\mathcal{R}}$ is the category of right modules over the square ring $\mathcal{R}$ and the morphisms preserving the structure.

Examples of modules

Over a ring R, seen as the square ring $(R \to 0 \to R)$

Since $\mathcal{R}_{\mathrm{ee}} = 0$, there are no bilinear operations, so the bracket is trivial. Also $H(r) \in \mathcal{R}_{\mathrm{ee}} = 0$, so each $m \mapsto m \cdot r$ is additive, and commutators vanish: M is an abelian group with an additive action of R. These are exactly the usual R-modules.

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Over $\mathbb{Z}_{\mathrm{nil}}$: groups of class ≤ 2

$m \cdot a$ is the a-fold sum of m, and the bracket is the x-fold sum of the commutator $n + m - n - m$. So $\mathrm{Mod}_{\mathbb{Z}_{\mathrm{nil}}} \cong \mathrm{Nil}_2(\mathrm{Grp})$.

Now, let's see how to characterise two-step nilpotency

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Thm. Let $\mathcal{V}$ be a variety of algebras. The following are equivalent:

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(ii) $\Rightarrow$ (i) is the "easy" part.

The "hard" direction: (i) $\Rightarrow$ (ii), the idea

1. The free object on one generator

$E = F_{\mathcal{V}}(1)$ is a **cogroup**. **This is where two-step nilpotency is used.**

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1. The free object on one generator

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2. The square ring

$\mathcal{R}_{\mathrm{e}} = \mathcal{V}(E, E)$ and $\mathcal{R}_{\mathrm{ee}} = \mathcal{V}(E, E \diamond E)$.

The "hard" direction: (i) ⇒ (ii), the idea

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$E = F_{\mathcal{V}}(1)$ is a **cogroup**. **This is where two-step nilpotency is used.**

2. The square ring

$\mathcal{R}_{\mathrm{e}} = \mathcal{V}(E, E)$ and $\mathcal{R}_{\mathrm{ee}} = \mathcal{V}(E, E \diamond E)$.

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$\mathcal{V}(E, -) : \mathcal{V} \to \mathrm{Mod}_{\mathcal{R}}$ is an equivalence of categories.

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The idea of the proof is similar to Freyd's, in the abelian case.

Talking about Freyd's theorem, is this compatible with it?

Yes, we can!

The abelian core is encoded in $\mathcal{R}$

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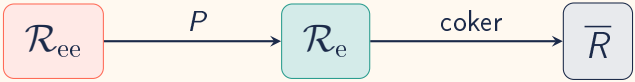

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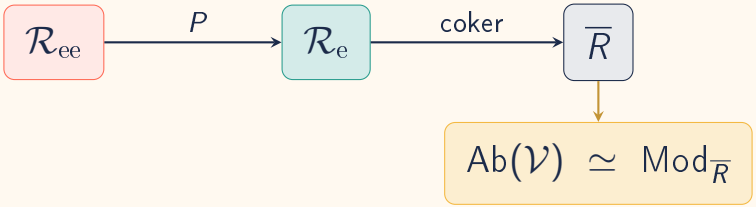

We can define the ring $\overline{R} = \mathsf{Coker}(\mathcal{R}_{\mathrm{ee}} \xrightarrow{P} \mathcal{R}_{\mathrm{e}})$, and it is the one characterising $\mathsf{Ab}(\mathcal{V})$ as $\mathsf{Mod}_{\overline{R}}$.

Why is this theorem really cool?

Applications

I like it

I like it

Tim likes it

Link with Theorem 7.6 of Deval–Hartl–Van der Linden

Question (DHVdL): given an abelian variety $\mathcal{A}$ with a bilinear product B, which semi-abelian varieties $\mathcal{V}$ have abelian core $\mathcal{A}$ and bilinear product B?

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Uniqueness

[Geenis, Hartl, Van der Linden, 2026]

By the main theorem, any such $\mathcal{V}$ is $\mathrm{Mod}_{\mathcal{R}'}$ for a square ring $\mathcal{R}'$ determined by $\mathcal{A}$ and B.

More consequences

- **Generators and relations.**
  Every term has a unique normal form: a sum of $m \cdot r$ and brackets $[m, n] \cdot x$. So a two-step nilpotent variety is fully described by $\mathcal{R}_{\mathrm{e}}$ and $\mathcal{R}_{\mathrm{ee}}$, like a polynomial is described by its linear and its bilinear part.

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- **Lower central series.**
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- **Group theory.**
  For any commutative ring R we get two-step nilpotent R-groups. This extends Hall's R-groups of class 2, which are only defined when R is 2-binomial.

A conclusion in alexandrines



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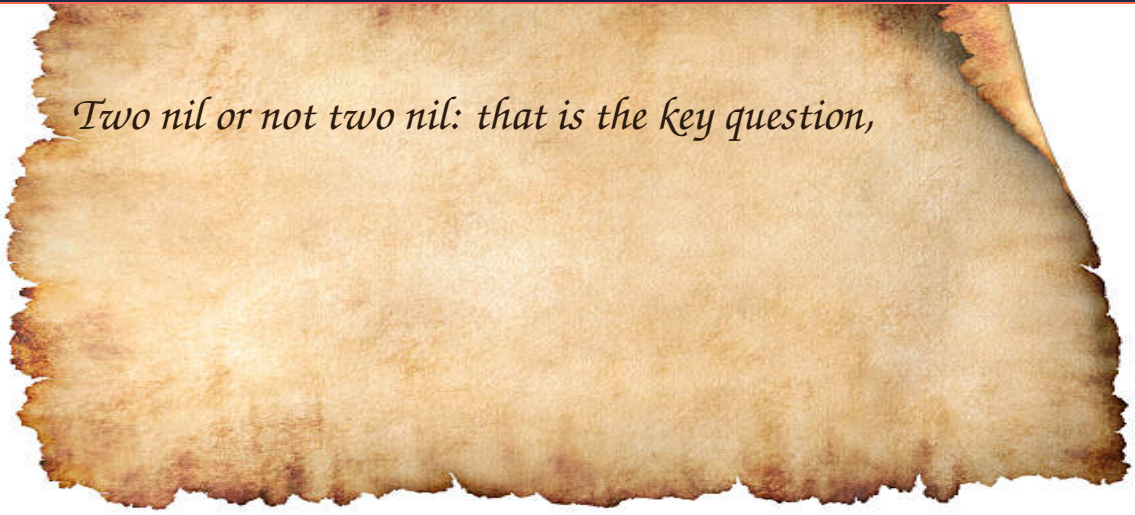

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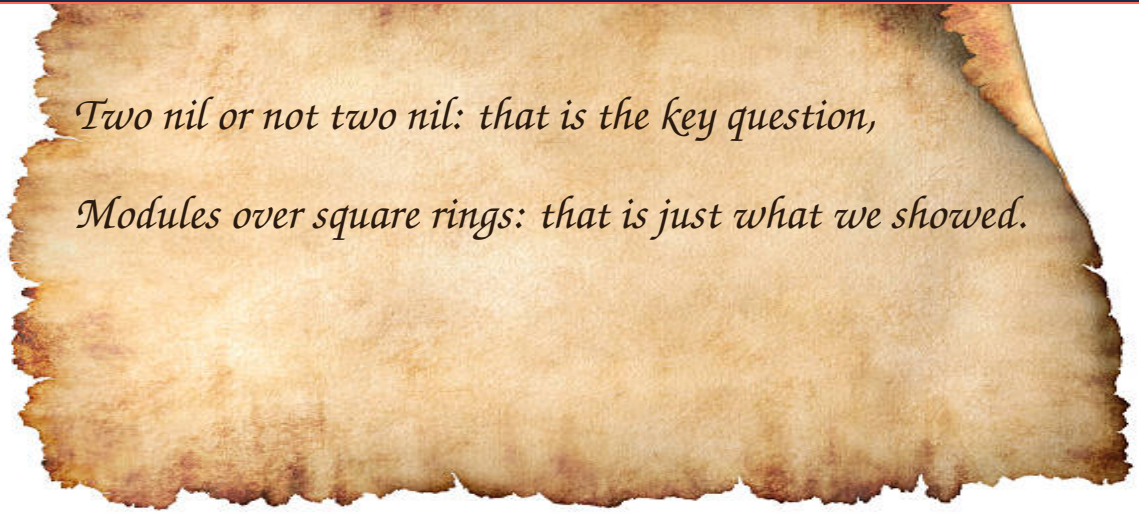

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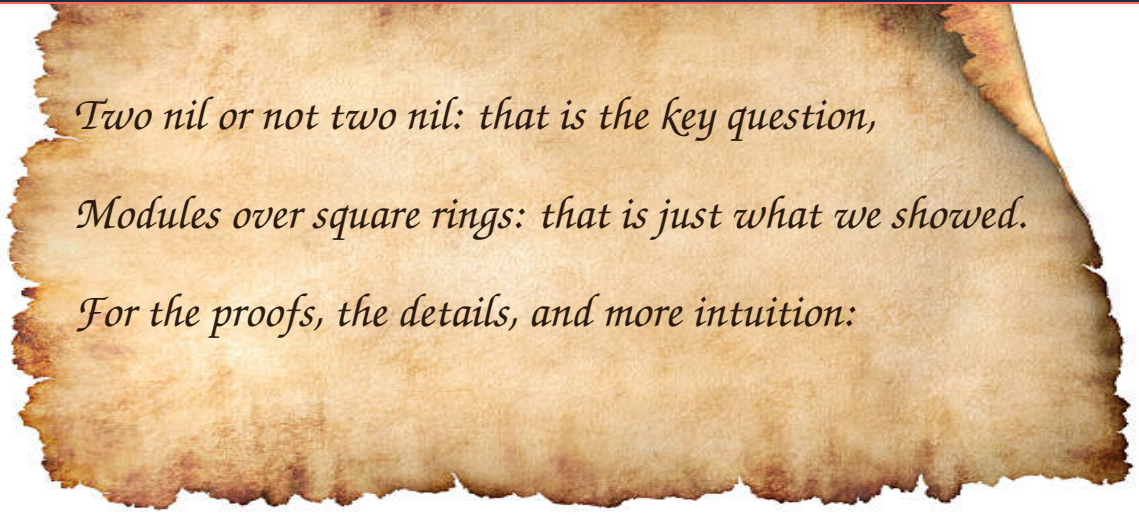

A conclusion in alexandrines

Two nil or not two nil: that is the key question,

Modules over square rings: that is just what we showed.

For the proofs, the details, and more intuition:

Just scan the QR code, our paper to download.



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Short answer:

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Short answer: I don't know. I assume our construction will not work for $n \geq 3$ because the arrow

$$X_1 + \cdots + X_n \to \prod_{k=1}^{n} X_1 + \cdots + \widehat{X_k} + \cdots + X_n$$

need not be a regular epimorphism. However, in the proofs, we use the fact that $X + Y \to X \times Y$ is one.

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1. Polynomial functor theory uses functors and natural transformations;
2. This object was introduced in the context of metastable homotopy theory;
3. **Square ringoids**: a square ring is a one-object square ringoid, i.e. a quadratic category. The theory of a two-step nilpotent variety is functorially one.



What are the links with homotopy theory?

Square rings come from homotopy theory

The motivating example is $\mathrm{End}(\Sigma\mathbb{RP}^2)$, the self-maps up to homotopy of the suspended real projective plane, with the sum induced by the suspension and the composition as product. It is a square ring by a result of Barratt.

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- It plays an important role in low-dimensional homotopy theory.

What is next? The Γ functor and exact sequences

Work in progress with Manfred Hartl and Tim Van der Linden

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- Use it to **extend exact homology sequences** in semi-abelian categories.

Same spirit as today: quadratic phenomena, made categorical.

Link with Hall's nilpotent R-groups of class 2?

Hall's definition [7], for a 2-binomial ring R

R commutative with $\binom{r}{2} = \frac{r(r-1)}{2}$ well defined. A **nilpotent R-group of class** 2 is a group M of class ≤ 2 with $(m, r) \mapsto m \cdot r$ such that

1. $m \cdot 1 = m, \quad m \cdot (r + s) = m \cdot r + m \cdot s, \quad m \cdot (rs) = (m \cdot r) \cdot s$;
2. $n + m \cdot r - n = (n + m - n) \cdot r$;
3. (Hall–Petresco) $m_1 \cdot r + \cdots + m_k \cdot r = (m_1 + \cdots + m_k) \cdot r + \sum_{i<j} (m_i + m_j - m_i - m_j) \cdot \binom{r}{2}$.

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Our answer

Every commutative ring R has a square ring R_{nil}; its modules are the **two-step nilpotent R-groups** (groups of class ≤ 2 for $R = \mathbb{Z}$), and Hall's groups when R is 2-binomial.

From Hall's groups to modules over R_{nil}

The square ring R_{nil}

$R_{\mathrm{nil}} = \left(R \xrightarrow{H_R} I_2 \xrightarrow{0} R\right)$, with I_2 the ideal generated by the elements $r(r-1)$,
$H_R(r) = r(r-1), \quad P = 0, \quad T = -1, \quad (r,s) \cdot x \cdot t = rsxt.$

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Conclusion

$\mathrm{Mod}_{R_{\mathrm{nil}}}$ is isomorphic to the category of Hall's nilpotent R-groups of class 2, and $I_2 = 2R$ then, since $r(r-1) = 2\binom{r}{2}$.