Some categorical properties in terms of operads

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Operads and their algebras

A *symmetric sequence* is a sequence $\{\mathcal{M}(n)\}$ such that $\mathcal{M}(n)$ is a K-linear S_n-representation.

$$(\mathcal{M}\otimes\mathcal{N})(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{S_i\times S_j}^{S_n} \mathcal{M}(i)\otimes\mathcal{N}(j),$$

$$(\mathcal{M}\circ\mathcal{N})(n) = \bigoplus_{k=0}^{+\infty} \mathcal{M}(k)\otimes_{S_k}(\mathcal{N}^{\otimes k})(n).$$

$$\mathcal{I} = (0, K, 0, 0\dots)$$

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We have a monoidal structure $(\mathsf{SymSeq}, \circ, \mathcal{I})$

An *operad* is a monoid in $(\mathsf{SymSeq}, \circ, \mathcal{I})$
A symmetric sequence $\mathcal{P}$ with

- A composition map $\gamma\colon \mathcal{P}\circ\mathcal{P}\to\mathcal{P}$
- A unit map $\eta\colon \mathcal{I}\to\mathcal{P}$

$$1_{\mathcal{P}} \coloneqq \eta(1) \in \mathcal{P}(1)$$

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An algebra over $\mathcal{P}$ is an algebra for the monoid.
A vector space A together with a linear map

$$\theta\colon \bigoplus_{n\geq 0} \mathcal{P}(n)\otimes A^{\otimes n} \to A$$

$$\mu\otimes a_1\otimes\cdots\otimes a_n \mapsto \mu(a_1,\dots,a_n)$$

If $\mathcal{P}(0) = 0$ the category of algebras over $\mathcal{P}$, $\mathcal{P}$-alg, is semi-abelian.

Categorical aspects

- The category Op fails to be pointed: $\mathcal{I} \neq \mathbb{0}$
- We define PsOp the slice category $\mathsf{Op}/\mathcal{I}$.

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2 Categorical aspects of operads

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A pseudo-operad is an augmented operad:

$$\varepsilon : \mathcal{P} \to \mathcal{I}$$

It can also be defined as a symmetric sequence $\mathcal{M}$ equipped with partial composition maps

$$\circ_i : \mathcal{M}(n) \otimes \mathcal{M}(m) \to \mathcal{M}(n+m-1)$$

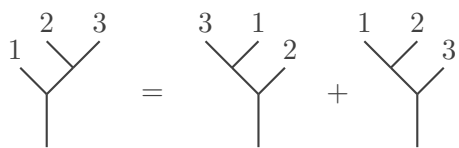
A pseudo-operad can be seen as an operad without a unit, and it relates to an operad just like a ring without a unit relates to a ring with unit.

- PsOp is semi-abelian and algebraically coherent.

Normality of Higgins I

- Normality of Higgins condition, NH, refers to whether the Higgins commutator of two normal subobjects is always normal.
- An example is the category of Lie algebras. A, B normal subobjects of X.

$$[x, [a, b]] = [[b, x], a] + [[x, a], b]$$



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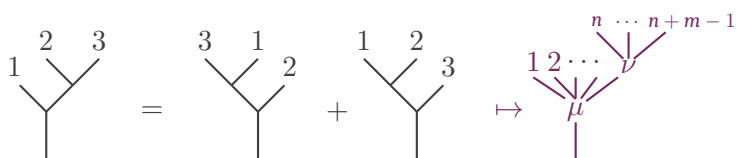
$$[x,[a,b]] = [[b,x],a] + [[x,a],b] \mapsto \mu(x_1,\ldots,x_{n-1},\nu(c_1,\ldots,c_m)) = \ldots$$

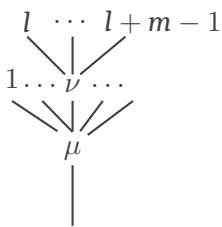


Normality of Higgins II

$n, m \geq 2, l \in [n]$

$$I_l = \{l, l+1, \ldots, l+m-1\} \subseteq [n+m-1]$$

$W_l(n, m)$ the space generated by trees of $n + m - 1$ leaves satisfying that in each principal branch we have at least one label of I_l.



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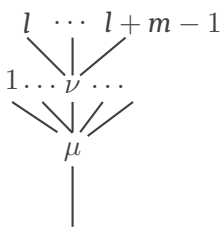

Characterization

The category of algebras over $\mathcal{P}$ satisfies Normality of Higgins commutators if and only if for all $\mu \in \mathcal{P}(n)$, $\nu \in \mathcal{P}(m)$ with $n, m \geq 2$ and $1 \leq l \leq n$:

$$\mu \circ_l \nu \in W_l(n, m)$$

Smith is Huq

- Smith is Huq condition, SH, can be characterized by the condition that

$$[A, B, X] \leq [A, B]^{\mathsf{Huq}}$$

Characterization

The category of algebras over an operad $\mathcal{P}$ satisfies the condition Smith is Huq if and only if the operad $\mathcal{P}$ is generated by binary operations.

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Characterization

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Sketch of the proof...

$$\mu(a, b, x) \in [A, B, X] \leq [A, B]^{\mathsf{Huq}} \Rightarrow \text{we can decompose } \mu.$$

$$\mu(a, b, x) = \sum (\nu_1 \circ_i \nu_2)(a, b, x) \in [A, B]^{\mathsf{Huq}}$$

Characterization

The category of algebras over an operad $\mathcal{P}$ is algebraically coherent if and only if it satisfies both NH and SH.

Sketch of the proof...

$$\text{AC} \Rightarrow \text{NH} + \text{SH}$$

$$\text{only binary operations} \Rightarrow (\text{AC} \Leftrightarrow \text{NH})$$

Algebraic Coherence II

The *enveloping algebra* of a $\mathcal{P}$-algebra B is an associative algebra, U_B, spanned by

$$\mu(b_1, \ldots, b_{i-1}, -, b_{i+1}, \ldots b_n)$$

The *enveloping operad* of a $\mathcal{P}$-algebra B is an operad $\mathcal{P}_B$ spanned in arity n by:

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The points over an object B are exactly the algebras over $\overline{\mathcal{P}}_B$.

The category of $\mathcal{P}$-algebras is algebraically coherent if and only if for all B the morphism

$$\mathcal{P} \circ U_B \to \overline{\mathcal{P}}_B$$

is surjective.

Local algebraic cartesian closedness

- The LACC property is characterized by the condition that the functor

$$\mathrm{Pt}_B(\mathcal{C}) \to \mathcal{C} \quad \mapsto \quad \overline{\mathcal{P}}_B\text{-alg} \to \mathcal{P}\text{-alg}$$

  has a right adjoint.

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- This implies a condition on the dimension of our operad.

$$\dim \mathcal{P}(n) = (n-1)! \dim \mathcal{P}(2)^{n-1}$$

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4 Bibliography

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Obrigada!