

When measurable spaces don't have enough points

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Abstract

In topology, a space is called sober if every irreducible closed subset is the closure of a unique point. One can express this concept in terms of the lower Vietoris monad, which assigns to a topological space X the space HX of its closed subsets. Given a space X , we can form a parallel pair $HX \rightarrow HHX$ using the unit of the monad, and the equalizer of this pair is precisely the set of irreducible closed subsets of X . The space X is sober if and only if it is an equalizer for this pair. One can instance the same construction in different contexts, and obtain analogous notions of "sobriety". For the Giry monad on measurable spaces, the equivalent of an irreducible closed set is a so-called "extremal" or "zero-one" measure. Just as an irreducible closed set cannot be written as a nontrivial union, an extremal measure cannot be written as a nontrivial convex combination. A measurable space is then sober if and only if every extremal measure is a Dirac delta at a unique point. Several measurable spaces used in mathematics fail to be sober, and have important nontrivial extremal measures. Examples are ergodic measures in dynamical systems, as well as measures arising from the zero-one laws of probability theory. These objects, despite being very useful in practice, are often described as being "singular", or even "badly behaved". Our categorical treatment, which parallels the one of topology, can give a systematic and structural understanding of these seemingly counterintuitive objects.

Joint work with Sean Moss. Relevant preprint: <https://arxiv.org/abs/2204.07003>.