When are exact categories co-exact?

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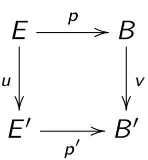
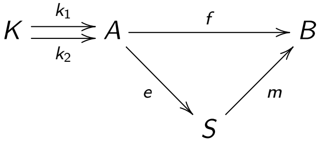
- The category $\mathbf{CHaus}_*$ of pointed compact Hausdorff topological spaces is co-semi-abelian:
  Theorem 1.3 of Frances Borceux and Maria Manuel Clementino. "On toposes, algebraic theories, semi-abelian categories and compact Hausdorff spaces". In: *Theory and Applications of Categories* 43.11 (2025), pp. 363–381
- In discussions around the above result Graham Manual suggested the question of whether the above fact could be a consequence of **CHaus** being a pretopos.

Definition

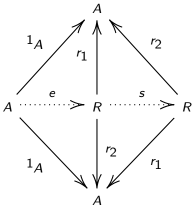
A category is a regular [Bar71] if it is finitely complete (admits all finite limits), regular epimorphisms are stable under pullback, and coequalizers of kernel pairs exist in it.

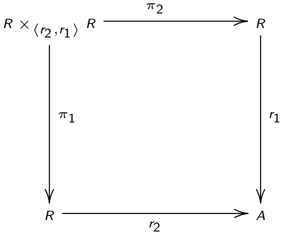


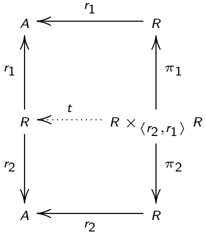


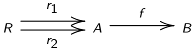
Definition

A category is a Barr-exact [Bar71] if it is regular and (the projections) of each (internal) equivalence relations form the kernel pair of some morphism.





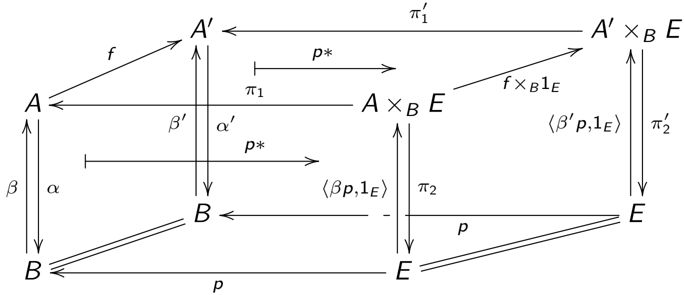




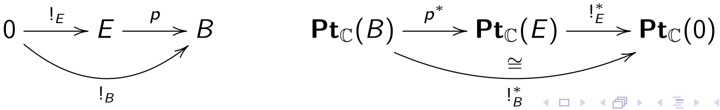
Definition

A category $\mathbb{C}$ is a protomodular [Bou91] if each change functor of the fibration of points reflects isomorphisms.

$$E \xrightarrow{p} B \qquad\qquad \mathbf{Pt}_{\mathbb{C}}(B) \xrightarrow{p^*} \mathbf{Pt}_{\mathbb{C}}(E)$$



If $\mathbb{C}$ has an initial object, then $\mathbb{C}$ is protomodular if and only if the change of base along morphisms of the form $!_B : 0 \to B$ reflect isomorphisms.



Definition

A category $\mathbb{C}$ is semi-abelian if it pointed, exact, protomodular, and has binary coproducts.

Examples include:

- The category of groups;
- Every sub-variety of non-associative algebras over a ring (including associative, Lie and Leibniz);
- Abelian categories;
- Cocommutative Hopf algebras [GSV19];
- $\mathbf{CHaus}_*^{\mathsf{op}}$ [BC25];
- $\mathbb{C}_*^{\mathsf{op}}$ where $\mathbb{C}_*$ is the category of pointed objects in a topos;
- $\mathbf{Pt}_{\mathbb{C}}(B)$ where $\mathbb{C}$ is semi-abelian;
- $\mathbb{C}^{\mathbb{X}}$ where $\mathbb{C}$ is semi-abelian.

Definition

A category $\mathbb{C}$ is ideally exact [Jan24] if is exact, protomodular, admits finite coproducts and the unique morphism from $0 \to 1$ is a regular epimorphism.

Examples include:

- semi-abelian categories;
- unital algebras over a ring;
- $(B \downarrow \mathbb{C})$ where $\mathbb{C}$ is ideally exact;

Note that if $\mathbb{C}$ is ideally exact then for each B in $\mathbb{C}$ the category $\mathbf{Pt}_{\mathbb{C}}(B)$ is semi-abelian.

Definition

A category is extensive if it has finite coproducts and each functor

$$
\begin{array}{ccc}
(\mathbb{C} \downarrow A) \times (\mathbb{C} \downarrow B) & \longrightarrow & (\mathbb{C} \downarrow A + B) \\
((X, f), (Y, g)) & \longmapsto & (X + Y, f + g) \\
\Big\downarrow{\scriptstyle (u,v)} & \longmapsto & \Big\downarrow{\scriptstyle u+v} \\
((X', f'), (Y', g')) & \longmapsto & (X' + Y', f + g')
\end{array}
\qquad (\#)
$$

is an equivalence of categories. This is equivalent to requiring that is has finite coproducts, pullbacks of coproduct inclusions along arbitrary morphisms exist, and for each diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{i} & C & \xleftarrow{j} & B \\
\Big\downarrow{\scriptstyle f} & (3) & \Big\downarrow{\scriptstyle h} & (4) & \Big\downarrow{\scriptstyle g} \\
A' & \xrightarrow[i']{} & C & \xleftarrow[j']{} & B'
\end{array}
\qquad
\begin{array}{c} (1) \\ \\ (2) \end{array}
$$

in which (2) is a coproduct (diagram), (1) is a coproduct if and only if (3) and (4) are pullbacks.

Definition

A category is a pretopos if it is Barr-exact and extensive.

Examples include:

- Toposes;
- **CHaus**;
- The category $\mathbf{Set}_\kappa$ of sets of cardinality bounded by κ;
- $(\mathbb{C} \downarrow B)$ where $\mathbb{C}$ is a pretopos;
- $\mathbb{C}^{\mathbb{X}}$ where $\mathbb{C}$ is a pretopos.

Proposition

If $\mathbb{C}$ be a pretopos, then

1. *pushouts of monomorphisms along arbitrary morphisms exist and form pullbacks in* $\mathbb{C}$*;*
2. *monomorphisms are stable under pushout in* $\mathbb{C}$ *i.e. "*$\mathbb{C}$ *is coregular minus the existence of all finite colimits;"*
3. *co-reflexive-relations are effective co-equivalence-relations in* $\mathbb{C}$*;*
4. $\mathbb{C}$ *is co-exact as soon as* $\mathbb{C}$ *admits pushouts of (regular) epimorphisms along (regular) epimorphisms;*
5. $\mathbb{C}$ *is co-protomodular;*
6. $\mathbb{C}$ *is co-ideally-exact (provided the above mentioned pushouts exist).*

Definition

A category $\mathbb{C}$ is additive if admits finite products and is enriched in abelian groups, which amounts to requiring that its hom-sets are equipped with an abelian group structures which are bilinear with respect to composition.

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C \xrightarrow{i} D \qquad\qquad i(g+h)f = igf + ihf$$

Definition

A category is abelian if it is additive and exact.

Recall:

Proposition

If $\mathbb{C}$ is an abelian category, then $\mathbb{C}^{op}$ is an abelian category (and in particular $\mathbb{C}$ is co-exact).

Condition 1

- (a) *finite coproducts exist in* $\mathbb{C}$*;*
- (b) *binary coproducts of pullback squares are pullback squares in* $\mathbb{C}$*;*
- (c) *For each monomorphism* $m : S \to A$ *in* $\mathbb{C}$ *the diagram*

$$\begin{array}{ccc} S+S & \xrightarrow{m+1_S} & A+S \\ {\scriptstyle [1_S,1_S]}\downarrow & & \downarrow{\scriptstyle [1_A,m]} \\ S & \xrightarrow[m]{} & A \end{array}$$

  is a pullback;
- (d) *for each regular epimorphism* $p : E \to B$ *in* $\mathbb{C}$ *the diagram*

$$\begin{array}{ccc} E+E & \xrightarrow{p+p} & B+B \\ {\scriptstyle [1_E,1_E]}\downarrow & & \downarrow{\scriptstyle [1_B,1_B]} \\ E & \xrightarrow[p]{} & B \end{array}$$

  is a feeble pullback (i.e. the unique morphism into the pullback is a regular epimorphism).

Proposition

If $\mathbb{C}$ is an exact category satisfying Condition 1, then

1. *pushouts of monomorphisms exist in $\mathbb{C}$ and are pullbacks;*
2. *monomorphisms are stable under pushout in $\mathbb{C}$;*
3. *monomorphisms and regular monomorphisms conincide in $\mathbb{C}$ and $\mathbb{C}$ is balanced;*
4. *co-reflexive-relations in $\mathbb{C}$ are effective co-equivalence relations;*
5. *$\mathbb{C}$ is co-exact as soon as it admits pushouts of (regular) epimorphisms along (regular) epimorphisms;*
6. *$\mathbb{C}$ is co-protomodular.*

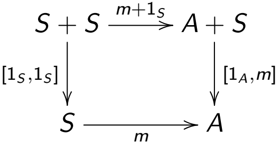
a. finite coproducts exist in $\mathbb{C}$;

Additive	Lextensive
binary products are coproducts	by assumption

b. binary coproducts of pullback squares are pullback squares in $\mathbb{C}$;

Additive	Lextensive
products of pullbacks are pullbacks	equivalences of categories preserve limits

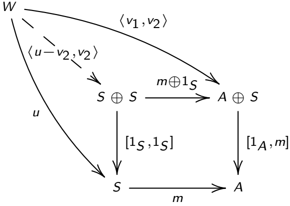
- For each monomorphism $m : S \to A$ in $\mathbb{C}$ the diagram



is a pullback;

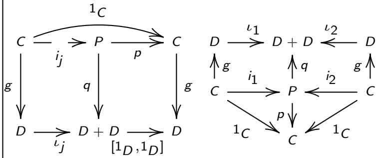
Additive	Lextensive

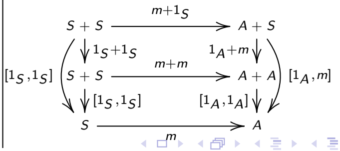
Additive



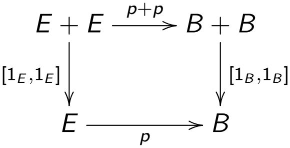
$(v_1 + mv_2 = mu) \Rightarrow (v_1 = m(u - v_2))$

Lextensive

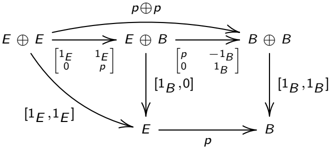
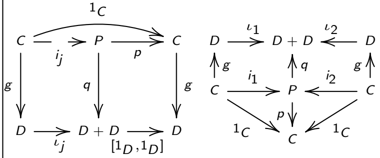




- for each regular epimorphism $p : E \to B$ in $\mathbb{C}$ the diagram



is a feeble pullback.

Additive	Lextensive
	

$$\begin{bmatrix} p & -1_B \\ 0 & 1_B \end{bmatrix} = \begin{bmatrix} 1_B & -1_B \\ 0 & 1_B \end{bmatrix} (p \oplus 1_B)$$

$$\begin{bmatrix} 1_E & 1_E \\ 0 & p \end{bmatrix} = (1 \oplus p) \begin{bmatrix} 1_E & 1_E \\ 0 & 1_E \end{bmatrix}$$

Lemma

If

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{f'} & B' \end{array}$$

is a feeble pullback and α and f are jointly monomorphic, then it is a pullback.

Lemma

For a diagram

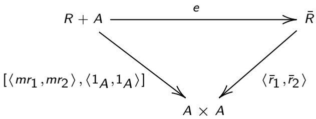
$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle\alpha}\big\downarrow & \boxed{1} & \big\downarrow{\scriptstyle\beta} & \boxed{2} & \big\downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

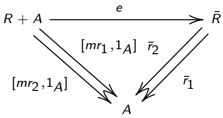
in a regular category $\mathbb{C}$ we have:

- (a) *if $\boxed{1}$ and $\boxed{2}$ are feeble pullbacks, then so is $\boxed{1}\boxed{2}$;*
- (b) *if f' is a regular epimorphism and $\boxed{1}\boxed{2}$ is a feeble pullback, then so is $\boxed{2}$;*

Lemma

Let $\mathbb{C}$ be a regular category satisfying Condition 1 let $m : S \to A$ be a monomorphism and let (R, r_1, r_2) be a equivalence relation on S in $\mathbb{C}$. The reflexive closure of (R, mr_1, mr_2) is an equivalence relation.





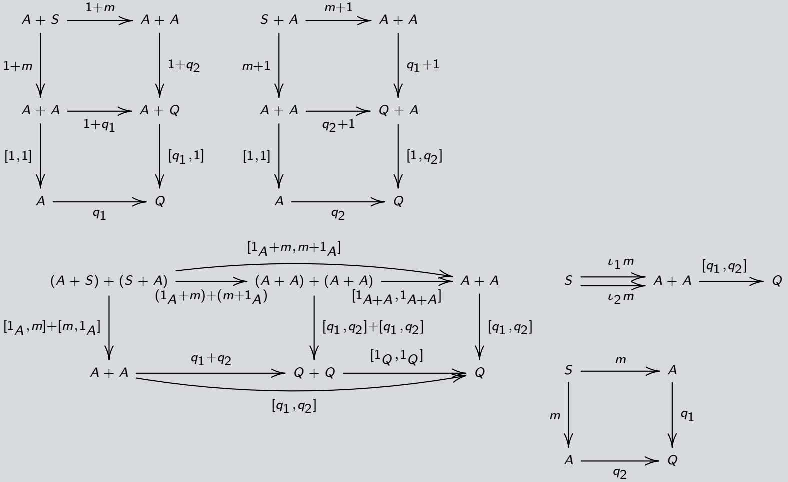
$$[\langle mr_1, mr_2\rangle, \langle 1_A, 1_A\rangle] = \langle [mr_1, 1_A], [mr_2, 1_A]\rangle$$

Proposition

Let $\mathbb{C}$ be an exact category satisfying Condition 1. Every co-reflexive-relation in $\mathbb{C}$ is an effective co-equivalence-relation.

Proof.

Suppose that $[q_1, q_2] : A + A \to Q$ is an epimorphism and $e : Q \to A$ is a morphism such that $e[q_1, q_2] = [1_A, 1_A]$. Let $m : S \to A$ be equalizer of q_1 and q_2.



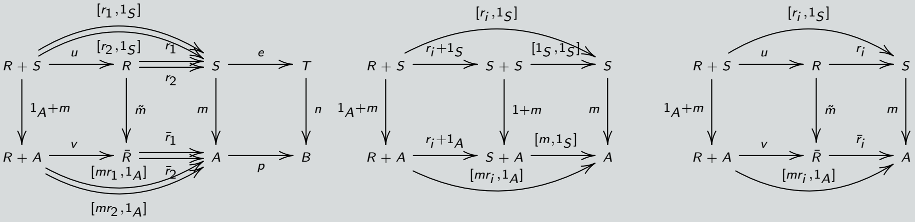
□

Proposition

Let $\mathbb{C}$ be an exact category satisfying Condition 1. The category $\mathbb{C}$ admits pushouts of monomorphisms along regular epimorphisms, which are pullbacks. Monomorphisms are stable under pushout along regular epimorphisms in $\mathbb{C}$.

Proof.

Let $e : S \to T$ be a regular epimorphism and let $m : S \to A$ be a monomorphism.



□

Proposition

Let $\mathbb{C}$ be a category satisfying Condition 1. $\mathbb{C}$ admits pushouts of monomorphisms along coproduct inclusions, which are pullbacks. Monomorphisms are stable under pushout along coproduct inclusions in $\mathbb{C}$.

Proof.

$$
\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ A & \xrightarrow[1_A]{} & A \end{array}
\quad + \quad
\begin{array}{ccc} S & \xrightarrow{n} & B \\ {\scriptstyle 1_S}\downarrow & & \downarrow{\scriptstyle 1_B} \\ S & \xrightarrow[n]{} & B \end{array}
\quad = \quad
\begin{array}{ccc} S & \xrightarrow{n} & B \\ {\scriptstyle \iota_2}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ A+S & \xrightarrow[1+n]{} & A+B \end{array}
$$

□

Proposition

Let $\mathbb{C}$ be an exact category satisfying Condition 1. The pushout of a monomorphism along an arbitrary morphism exists and produces a pullback in $\mathbb{C}$. Monomorphisms are pushout stable in $\mathbb{C}$.

Proof.

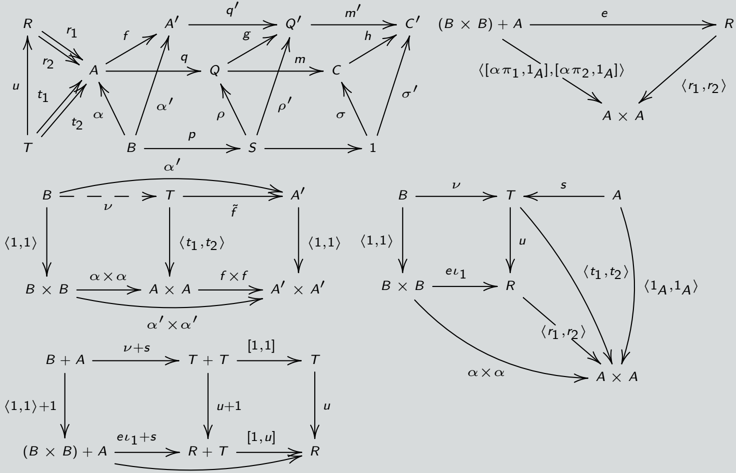
$$A \xrightarrow{\iota_1} A + B \xrightarrow{[f,1]} B, \qquad [f,1] \circ \iota_1 = f$$

□

Theorem

Let $\mathbb{C}$ be an exact category satisfying Condition 1. $\mathbb{C}$ is coprotomodular.

Proof.



□

Theorem

The opposite category of a pretopos with finite colimits is arithmetical and ideally exact.

[Bar71] Michael Barr. "Exact Categories". In: *in: Lecture Notes in Mathematics* 236 (1971), pp. 1–120.

[BC25] Frances Borceux and Maria Manuel Clementino. "On toposes, algebraic theories, semi-abelian categories and compact Hausdorff spaces". In: *Theory and Applications of Categories* 43.11 (2025), pp. 363–381.

[Bou91] Dominique Bourn. "Normalization equivalence, kernel equivalence and affine categories". In: *Category theory (Como, 1990)*. Vol. 1488. Lecture Notes in Mathematics. Springer, Berlin, 1991, pp. 43–62. DOI: 10.1007/BFb0084212. URL: https://doi.org/10.1007/BFb0084212.

[GSV19] Marino Gran, Florence Sterck, and Joost Vercruysse. "A semi-abelian extension of a theorem by Takeuchi". In: *Journal of Pure and Applied Algebra* (2019). ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2019.01.004. URL: http://www.sciencedirect.com/science/article/pii/S002240491930012X.

[Jan24] George Janelidze. "Ideally exact categories". In: *Theory and Applications of Categories* 41 (2024), Paper No. 11, 414–425. ISSN: 1201-561X.