Drazin Inverses in (Dagger) Categories
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JS PL (he/him), joint work with Robin Cockett and Priyaa Varshinee Srinivasan

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Quick Hello!

- Full Name: Jean-Simon Pacaud Lemay, please feel free to call me **JS**
- I'm from Québec, Canada
- PhD: **University of Oxford**
- I'm a assistant professor at **Macquarie University** and part of the **Australian Category Theory Centre**
- I'm a category theorist, and I study:

  • **Differential Categories**
  • **Tangent Categories**
  • Differential Geometry, Algebraic Geometry, Differential Algebras
  • Traced Monoidal Categories
  • Restriction Categories
  • Other stuff...

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  • Restriction Categories
  • Other stuff... like **Generalized Inverses**

Today's Main References

Today's story is about a special kind of generalized inverses called **Drazin Inverses** (and also a bit about **Moore-Penrose Inverses**) through the lens of category theory. This has become a new research theme for us, which I've been very excited about, and that has also seemingly picked up interest throughout the community (hence why I'm talking to you about it today!)

- **Drazin Inverses in Categories**, TAC 2025

  `http://www.tac.mta.ca/tac/volumes/43/14/43-14abs.html`

- **Dagger-Drazin Inverses**, QPL 2025

  `https://arxiv.org/pdf/2502.05306`

- **Moore-Penrose Dagger Categories**, QPL 2023

  `https://cgi.cse.unsw.edu.au/~eptcs/paper.cgi?QPL2023.10`

Drazin Inverses in Ring Theory

In a **ring** R, a **Drazin inverse** of $x \in R$ is a $x^D \in R$ such that:

- **[D.1]** There is a $k \in \mathbb{N}$ such that $x^{k+1}x^D = x^k$
- **[D.2]** $x^D x x^D = x^D$
- **[D.3]** $x^D x = x x^D$

While a Drazin inverse may not always exist, if a Drazin inverse exists then it is unique, so we may speak of *the* Drazin inverse.

The "inverse" part in the term "Drazin inverse" is justified since it is a generalization of the usual notion of inverse. In a **ring** R, if $x \in R$ is invertible, then x^{-1} is the Drazin inverse of x.

The term Drazin inverse is named after Michael P. Drazin (1929 – still alive!), who originally introduced the concept of Drazin inverses in rings under the name "pseudo-inverse"

Michael P. Drazin **Pseudo-inverses in associative rings and semigroups.** (1958)

Drazin Inverses in Ring Theory

In a **semigroup** R, a **Drazin inverse** of $x \in R$ is a $x^D \in R$ such that:

- **[D.1]** There is a $k \in \mathbb{N}$ such that $x^{k+1}x^D = x^k$
- **[D.2]** $x^D x x^D = x^D$
- **[D.3]** $x^D x = x x^D$

While a Drazin inverse may not always exist, if a Drazin inverse exists then it is unique, so we may speak of *the* Drazin inverse.

The "inverse" part in the term "Drazin inverse" is justified since it is a generalization of the usual notion of inverse. In a **monoid** R, if $x \in R$ is invertible, then x^{-1} is the Drazin inverse of x.

The term Drazin inverse is named after Michael P. Drazin (1929 – still alive!), who originally introduced the concept of Drazin inverses in rings under the name "pseudo-inverse"

Michael P. Drazin **Pseudo-inverses in associative rings and semigroups.** (1958)

Drazin Inverses all around

Drazin inverses have an extensive literature and active area of research:

- Studied in-depth in ring theory and semigroup theory
- Connected to **strong π-regularity**
- Connected to Fitting's results (Fitting's Lemma or Fitting's Decomposition result)
- Studied in matrix theory since every complex square matrix has a Drazin inverse. So the Drazin inverse has many application and is a very useful tool for computations

But what about Drazin inverses in category theory?

For any object A in a category $\mathbb{X}$, the homset $\mathbb{X}(A, A)$ of endomorphisms of type A is a monoid with respect to composition. As such, we may consider Drazin inverses in $\mathbb{X}(A, A)$, or in other words, we may talk about Drazin inverses of endomorphisms in an arbitrary category.

However not much had been previously done with Drazin inverses in category theory!

To the best of our knowledge, the only previous discussion of Drazin inverses in category theory appears in a section of this paper:

R. Puystjens & D. W. Robinson **Generalized Inverses of Morphisms with Kernels**. (1987)

where they provide an existence property of Drazin inverses in an additive/Abelian category.

At some point after we wrote this paper:

R. Cockett & J.-S. P. Lemay **Moore-Penrose Dagger Categories**. (2023)

Robin became fascinated by Drazin inverses! We realized lots can be said about Drazin inverses using a categorical point of view.

Quick summary of our paper

The purpose of our paper was to develop Drazin inverses from a categorical perspective. We both review the ring/semigroup theory stuff, and also provide novel results.

- Drazin inverses in a category
- Consider Drazin categories and many examples
- How Drazin inverses behave well with well-known categorical constructions
- A 2-categorical perspective on Drazin inverses (rank!)
- Relating Drazin inverses to idempotent splitting
- Relate Drazin inverse to eventual image duality:

  T. Leinster **The Eventual Image**. (2022)

- Drazin inverses in additive/Abelian categories, recapturing Fitting's results
- Generalize the notion of Drazin inverses to that of **Drazin opposing pairs**.

We'll go through some of this today. But any of this sounds interesting please see our paper:

`https://arxiv.org/pdf/2402.18226.pdf`

Drazin Inverses in a Category

Definition

In a category $\mathbb{X}$, a **Drazin inverse** of $x : A \to A$ is an endomorphism $x^D : A \to A$ such that[a]:

- **[D.1]** There is a $k \in \mathbb{N}$ such that $x^{k+1}x^D = x^k$
- **[D.2]** $x^D x x^D = x^D$
- **[D.3]** $x^D x = x x^D$

If $x : A \to A$ has a Drazin inverse $x^D : A \to A$, we say that x is **Drazin**, and call the least k such that $x^{k+1}x^D = x^k$ the **Drazin index of** x, which we denote by $\mathrm{ind}(x) = k$.

[a] Composition is written in *diagrammatic order* – not that it will matter much...

Proposition

In a category $\mathbb{X}$, if $x : A \to A$ has a Drazin inverse, then it is unique.

More properties of Drazin inverses later after examples...

Drazin Categories **NEW**

Definition

A **Drazin category** is a category such that every endomorphism has a Drazin inverse.

It is always possible to construct a Drazin category from any category by considering the full subcategory determined by the objects whose every endomorphism is Drazin.

Definition

In a category $\mathbb{X}$, an object A is a **Drazin object** if every endomorphism $x : A \to A$ is Drazin. Let $\mathsf{D}(\mathbb{X})$ be the full subcategory of Drazin objects of $\mathbb{X}$.

Lemma

For any category $\mathbb{X}$, $\mathsf{D}(\mathbb{X})$ *is a Drazin category. Moreover,* $\mathbb{X}$ *is Drazin if and only if* $\mathsf{D}(\mathbb{X}) = \mathbb{X}$.

Drazin Inverse Example: Matrices

Example

Let F be field and $\mathsf{MAT}(F)$ be the category of matrices over k, that is, the category whose objects are natural numbers $n \in \mathbb{N}$ and where a map $A : n \to m$ is an $n \times m$ F-matrix. Composition given by matrix multiplication and the identity on n is the n-dimensional identity matrix. Endomorphisms in $\mathsf{MAT}(F)$ correspond precisely to square matrices: so an endomorphism $A : n \to n$ is an $n \times n$ square matrix A. Then $\mathsf{MAT}(F)$ is a Drazin category.

So for an $n \times n$ matrix A, to compute its Drazin inverse we first write it in the form:

$$A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}$$

for some invertible $n \times n$ matrix P, an invertible $m \times m$ matrix C (where $m \leq n$), and a nilpotent $n - m \times n - m$ matrix N (that is, $N^k = 0$ for some $k \in \mathbb{N}$).

$$A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

The Drazin index of A corresponds precisely to the **index** of A, which is the least $k \in \mathbb{N}$ such that $\mathsf{rank}(A^{k+1}) = \mathsf{rank}(A^k)$.

Drazin inverses of **complex** matrices are well studied and have many applications.

S. Campbell & C. Meyer. **Generalized inverses of linear transformations.**

Drazin Inverse Example: Modules

Example

Let R be a ring and let R-MOD be the category of (left) R-modules and R-linear morphisms between them. In general, R-MOD is not Drazin...

For example when $R = \mathbb{Z}$, the $\mathbb{Z}$-linear endomorphism $f : \mathbb{Z} \to \mathbb{Z}$ defined $f(x) = 2x$ does not have a Drazin inverse. Why?

If f had a Drazin inverse $f^D : \mathbb{Z} \to \mathbb{Z}$, by $\mathbb{Z}$-linearity, it must be of the form $f^D(x) = nx$ for some $n \in \mathbb{Z}$. Then by **[D.1]** we would have that for some $k \in \mathbb{N}$, $2^{k+1}nx = 2^k x$ for all $x \in \mathbb{Z}$. This would imply that $2n = 1$, but since $n \in \mathbb{Z}$, this is a contradiction.

So $\mathbb{Z}$-MOD is not Drazin.

Drazin Inverse Example: Modules

Example

While R-MOD may not always be Drazin, there are various characterizations of Drazin R-linear endomorphism!

Z. Weng **Class of drazin inverses in rings**. (2017)

In particular, an R-linear endomorphism $f : M \to M$ is Drazin if and only if

$$M = \mathrm{im}(f^k) \oplus \ker(f^k)$$

for some $k \geq 1$ and this decomposition is sometimes called **Fitting's decomposition**. In this case, f becomes an isomorphism on $\mathrm{im}(f^k)$, and its Drazin inverse is the inverse on this component.

Then an R-module M is said to satisfy **Fitting's Lemma** if every endomorphism has a Fitting's decomposition (or equivalently if R-MOD(M, M) is **strongly π-regular**).

E. P. Armendariz, J. W. Fisher, and R. L. Snider. **On injective and surjective endo-morphisms of finitely generated modules**. (1978)

As such, the Drazin objects in R-MOD are precisely the R-modules which satisfy Fitting's Lemma.

Drazin Inverse Example: Modules – Fitting Interlude

Hans Fitting was a German mathematician who died in 1938 – unexpectedly – at the young age of 31. His results (written in German) are now so fundamental that they are simply referred to as "Fitting's Lemma" and "Fitting's decomposition theorem".

H. Fitting. **Die theorie der automorphismenringe abelscher gruppen und ihr analogon bei nicht kommutativen gruppen.** (1933)

Fitting's Decomposition Theorem says that for every endomorphism of a **finite length** R-module gives a Fitting Decomposition. **Fitting's Lemma** says that every endomorphism of an *indecomposible* finite length module is either an isomorphism or a nilpotent.

Example

This implies that every R-linear endomorphism of a finite length R-module is Drazin, and thus finite length R-modules are Drazin. So the full subcategory of finite length R-modules is Drazin.

REMARK: While every finite length R-module is Drazin, there are modules which do not have finite length which are Drazin. Ex. $\mathbb{Q}$ seen as a $\mathbb{Z}$-module is Drazin but not of finite length!

Drazin Inverse Example: Finite Sets

Example

Let FinSET be the category of finite sets and functions between them. FinSET is Drazin.

When X is a finite set, one way of understanding the Drazin inverse of a function $f : X \to X$, is to consider the inclusion of subsets:

$$X \supseteq \mathrm{im}(f) \supseteq \mathrm{im}(f^2) \supseteq \ldots \supseteq \mathrm{im}(f^k) = \mathrm{im}(f^{k+1}) = \ldots$$

which must eventually stabilize after at most $k \leq |X|$ steps. Then f becomes an isomorphism on $\mathrm{im}(f^k)$. Then the Drazin inverse of f is:

$$f^D(x) = f|^{-1}_{\mathrm{im}(f^k)}(f^k(x))$$

Lemma

Every finite set enriched category is Drazin.

Drazin Inverse Example: Sets

Example

Let SET be the category of sets and functions between them.

SET is not Drazin. For example the successor function $s : \mathbb{N} \to \mathbb{N}$, $s(n) = n + 1$, does not have a Drazin inverse. Why?

Suppose that s had a Drazin inverse $s^D : \mathbb{N} \to \mathbb{N}$. By **[D.3]**, we would have that:

$$s^D(n) = s^D(s^n(0)) = s^n(s^D(0)) = s^D(0) + n$$

So $s^D(n) = s^D(0) + n$. Now if $\mathrm{ind}(s) = k$, by **[D.1]** we would have that:

$$k = s^k(0) = s^D(s^{k+1}(0)) = s^D(k+1) = s^D(0) + k + 1$$

This implies that $0 = s^D(0) + 1$ – which is a contradiction since $s^D(0) \in \mathbb{N}$.

But we may still ask what are the Drazin objects are...

Lemma

A set X is Drazin in SET *if and only if X is a finite set. Therefore* $\mathrm{D}(\mathrm{SET}) = \mathrm{FinSET}$.

Properties of Drazin Inverses

Now let's look at some properties of Drazin inverses.

WARNING about composition

Unfortunately, Drazin inverses do not necessarily play well with composition.

Even if x and y are Drazin, xy may not be Drazin...

And even if xy is Drazin, we might not have that $(xy)^D$ is equal to $y^D x^D$...

Strongly π-Regular

Definition

In a category $\mathbb{X}$, $x : A \to A$ is **strongly π-regular** if there exists endomorphisms $y : A \to A$ and $z : A \to A$, and $p, q \in \mathbb{N}$ such that $yx^{p+1} = x^p$ and $x^{q+1}z = x^q$.

Lemma

In a category $\mathbb{X}$, $x : A \to A$ is Drazin if and only if it x is strongly π-regular.

Strongly π-regular rings have been studied in-depth:

- E. P. Armendariz, J. W. Fisher, and R. L. Snider. **On injective and surjective endo-morphisms of finitely generated modules**. (1978)
- P. Ara. **Strongly π-regular rings have stable range one.** (1996)
- G. Azumaya. **Strongly π-regular rings.** (1954)
- M. F. Dischinger. **Sur les anneaux fortement π-regulier.** (1979)
- W. K. Nicholson. **Strongly clean rings and Fitting's Lemma.** (1999)

Isomorphisms and Drazin Index of 0

Lemma

In a category $\mathbb{X}$, $x : A \to A$ *is Drazin with* $\mathrm{ind}(x) = 0$ *if and only if* x *is an isomorphism. Explicitly:*

- *If* x *is an isomorphism, then it is Drazin where* $x^D = x^{-1}$ *and* $\mathrm{ind}(x) = 0$*;*
- *If* x *is Drazin and* $\mathrm{ind}(x) = 0$*, then* x *is an isomorphism where* $x^{-1} = x^D$*.*

In particular, the identity $1_A : A \to A$ *is Drazin and its own Drazin inverse,* $1_A^D = 1_A$*.*

Group Inverses and Drazin Index ≤ 1

Definition

In a category $\mathbb{X}$, a **group inverse** of $x : A \to A$ is an endomorphism $x^D : A \to A$ such that the following equalities hold:

- **[G.1]** $xx^D x = x$;
- **[G.2]** $x^D xx^D = x^D$;
- **[G.3]** $x^D x = xx^D$.

Lemma

In a category $\mathbb{X}$, $x : A \to A$ is Drazin with $\mathrm{ind}(x) \leq 1$ if and only if x has a group inverse. Explicitly:

- *If x has a group inverse x^D, then x is Drazin with the group inverse x^D being its Drazin inverse and $\mathrm{ind}(x) \leq 1$;*
- *If x is Drazin and $\mathrm{ind}(x) \leq 1$, then its Drazin inverse x^D is its group inverse.*

Drazin Inverse of a Drazin Inverse

Lemma

In a category $\mathbb{X}$, *let* $x : A \to A$ *be Drazin. Then:*

- x^D *is Drazin where* $x^{DD} := xx^Dx$ *and* $\mathrm{ind}(x^D) \leq 1$*;*
- x^{DD} *is Drazin where* $x^{DDD} = x^D$*;*
- *If* $\mathrm{ind}(x) \leq 1$*, then* $x^{DD} = x$*.*

Iteration

Lemma

In a category $\mathbb{X}$, if $x : A \to A$ is Drazin, then x^n is Drazin where $(x^n)^D = (x^D)^n$.

Lemma

In a category $\mathbb{X}$, $x : A \to A$ is Drazin if and only if there is a $k \in \mathbb{N}$ such that $x^{k+1} : A \to A$ is Drazin.

Absolute

We now turn our attention to other properties of Drazin inverse that have a more categorical flavour....

Proposition

Let $\mathsf{F} : \mathbb{X} \to \mathbb{Y}$ *be a functor and let* $x : A \to A$ *be Drazin in* $\mathbb{X}$*. Then* $\mathsf{F}(x) : \mathsf{F}(A) \to \mathsf{F}(A)$ *is Drazin in* $\mathbb{Y}$ *where* $\mathsf{F}(x)^D = \mathsf{F}(x^D)$ *and* $\mathrm{ind}\left(\mathsf{F}(x)\right) \leq \mathrm{ind}(x)$.

Corollary

If a category $\mathbb{X}$ *is equivalent to a category* $\mathbb{Y}$ *which is Drazin, then* $\mathbb{X}$ *is Drazin.*

Example

So for a field k, let k-FVEC be the category of finite dimensional k-vector spaces and k-linear maps between them. Since k-FVEC is equivalent to $\mathsf{MAT}(k)$, we get that k-FVEC is Drazin. (Another way to see this is that finite dimensional vector spaces are have finite length).

Commuting Squares

Proposition

In a category $\mathbb{X}$, let $x : A \to A$ and $y : B \to B$ be Drazin. If the diagram on the left commutes, then the diagram on the right commutes:

$$\begin{array}{ccc} A & \xrightarrow{x} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & B \end{array} \quad \Rightarrow \quad \begin{array}{ccc} A & \xrightarrow{x^D} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y^D]{} & B \end{array}$$

This is quite useful for constructing new Drazin categories:

- (Co)Slice categories;
- (Co)algebras of endofuctions;
- Chu Construction

Drazin Inverses and Idempotents

There is a deep connection between Drazin inverses and idempotents!

Lemma

Let $x : A \to A$ *be Drazin. Define the map* $e_x := x^D x : A \to A$ *(or equivalently by* **[D.3]** *as* $e_x = xx^D$*). Then* e_x *is an idempotent.*

Lemma

An idempotent $e : A \to A$ *is Drazin, its own Drazin inverse,* $e^D = e$*, and* $\mathrm{ind}(e) \leq 1$*. Moreover,* $\mathrm{ind}(e) = 0$ *if and only if* $e = 1_A$*.*

Drazin Inverses and Idempotent Splitting

Another way of understanding Drazin inverses is as isomorphisms in the **idempotent splitting**.

For a category $\mathbb{X}$, let $\mathsf{Split}(\mathbb{X})$ be its idempotent splitting. Recall:

- Objects are pairs (A, e) consisting of an object A and an idempotent $e : A \to A$
- A $f : (A, e) \to (B, e')$ is a map $f : A \to B$ such that $efe' = f$ (or equivalently $ef = f = fe'$).
- Composition same as in $\mathbb{X}$
- Identity maps are $1_{(A,e)} := e : (A, e) \to (A, e)$.

Theorem

$x : A \to A$ is Drazin in $\mathbb{X}$ if and only if there is an idempotent $e : A \to A$ such that for some $k \in \mathbb{N}$, $x^{k+1} : (A, e) \to (A, e)$ is an isomorphism in $\mathsf{Split}(\mathbb{X})$.

Proof.

For the $\Rightarrow$ take the idempotent e_x. Then $x^{k+1} : (A, e_x) \to (A, e_x)$ is an isomorphism in $\mathsf{Split}(\mathbb{X})$.

The $\Leftarrow$ direction requires more work. Briefly, if $x^{k+1} : (A, e) \to (A, e)$ is an isomorphism in $\mathsf{Split}(\mathbb{X})$ with inverse $v : (A, e) \to (A, e)$. Then the Drazin inverse of x is $x^D := vx^k = x^k v$. □

Quick Word about Eventual Image Duality and Drazin Inverses

We also looked at the relationship between Drazin inverses and Leinster's eventual image duality.

Briefly, an endomorphism $x : A \to A$ has an Eventual Image Duality if the diagram:

$$\dots \xrightarrow{\quad x \quad} A \xrightarrow{\quad x \quad} A \xrightarrow{\quad x \quad} A \xrightarrow{\quad x \quad} \dots$$

has both a limit and colimit, which are canonically isomorphic.

T. Leinster **The Eventual Image**. (2022)

Definition

$x : A \to A$ is **Drazin split** if it is Drazin and the induced idempotent $e_x := xx^D : A \to A$ splits.

Lemma

An endomorphism that is Drazin split has eventual image duality.

Quick Word about Expressive Rank

For a square matrix A (over a field), an intuitive way of finding its Drazin inverse is to iterate A until the rank does not change (which is always guaranteed to happen):

$$\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^{k+2}) = \dots$$

When this happens, one can reverse any later iterations and thus build a Drazin inverse. The same principle holds true for linear endomorphisms on a finite-dimensional vector space or endomorphisms on a finite set.

We made this procedure rigorous categorically, which involved making precise what is meant by "rank". In linear algebra, the rank of a matrix or a linear transformation is the dimension of its image space. So we generalized this in a category by associating every map to a natural number which represents its rank.

We express this as saying that a category has **expressive rank** if there is a *colax* functor into a 2-category which we call Rank (which is span of the poset of natural numbers).

Theorem

A category which has an expressive rank is Drazin.

Quick word about Fitting's Results in an Abelian Category

Theorem

In an Abelian category $\mathbb{X}$, $x : A \to A$ is Drazin if and only if there is some $k \in \mathbb{N}$ such that the canonical map:

$$\mathsf{im}(x^{k+1}) \oplus \mathsf{ker}(x^{k+1}) \to A$$

is an isomorphism.

There is much more to say about Drazin inverses in the additive setting!

- Generalizing Matrix Example in finite biproduct setting
- Core-Nilpotent Decomposition
- Kernel-Cokernel Coincidence, recapturing Robinson and Puystjens result/formula.

Drazin Inverses for Arbitrary Maps?

Arriving with categorical eyes to the subject of Drazin inverses it is natural to want to have a Drazin inverse not just for endomorphisms $x : A \to A$ but for an arbitrary map $f : A \to B$.

However, to have a Drazin inverse of a map $f : A \to B$, one really needs an *opposing* map $B \to A$ to allow for the iteration which is at the heart of the notion of a Drazin inverse.

It is natural to do so in a **dagger category**, which leads us to the notion of **dagger-Drazin inverses**. For the full story, see our QPL2025 paper:

`https://arxiv.org/pdf/2502.05306`

Dagger Categories

Definition

A **dagger** on a category $\mathbb{X}$ is a contravariant functor $(\_)^\dagger : \mathbb{X} \to \mathbb{X}$ which is the identity on objects and involutive. We refer to $f^\dagger$ as the **adjoint** of f.

A †-**category** is a category $\mathbb{X}$ equipped with a chosen dagger †.

Concretely, a †-category can be described as a category $\mathbb{X}$ where for each map $f : A \to B$, there is a chosen map of dual type $f^\dagger : B \to A$ such that:

$$1_A^\dagger = 1_A \qquad (fg)^\dagger = g^\dagger f^\dagger \qquad (f^\dagger)^\dagger = f$$

Dagger Categories - Examples

Example

MAT($\mathbb{K}$) is a $\dagger$-category whose dagger is the transpose operator T, $A^{\mathsf{T}}(i,j) = A(j,i)$.

Example

If $\mathbb{K}$ is an involutive field, with involution $x \mapsto \overline{x}$, then MAT($\mathbb{K}$) has another dagger this time given by the conjugate transpose operator $*$, $A^*(i,j) = \overline{A(j,i)}$.

Example

Let HILB be the category of (complex) Hilbert spaces and bounded linear operators. Then HILB is a $\dagger$-category where the dagger is given by the adjoint $*$, that is, for a bounded linear operator $f : H_1 \to H_2$, $f^* : H_2 \to H_1$ is the unique bounded linear operator such that:

$$\langle f(x) | y \rangle = \langle x | f^*(y) \rangle$$

Similarly FHILB, the subcategory of finite dimensional Hilbert spaces, is a $\dagger$-category with the same dagger. This dagger corresponds to the conjugate transpose on MAT($\mathbb{C}$).

Dagger Drazin Inverse

Definition

In a †-category, a **†-Drazin inverse** of a map $f : A \to B$ is a map of dual type $f^\partial : B \to A$ such that:

- **[D†.1]** There is a $k \in \mathbb{N}$ such that $(ff^\dagger)^k ff^\partial = (ff^\dagger)^k$ and $f^\partial f(f^\dagger f)^k = (f^\dagger f)^k$;
- **[D†.2]** $f^\partial ff^\partial = f^\partial$;
- **[D†.3]** $(ff^\partial)^\dagger = ff^\partial$;
- **[D†.4]** $(f^\partial f)^\dagger = f^\partial f$.

If f has a †-Drazin inverse, we say that f is **†-Drazin invertible** or simply **†-Drazin**.

We call the smallest natural number k such that **[D†.1]** holds the **†-Drazin index of f**, which we denote by $\mathrm{ind}^\partial(f)$.

A **†-Drazin category** is a †-category such that every map is †-Drazin.

Proposition

In a †-category, if a map has a †-Drazin inverse, then it is unique.

Relating Dagger Drazin Inverses to Drazin Inverses

Before we see examples of dagger Drazin inverses, let's see how they relate to Drazin inverses.

Drazin Inverses and Daggers

Lemma

In a †-category, an endomorphism x is Drazin if and only if $x^\dagger$ is Drazin. Moreover, if x is Drazin then ${x^\dagger}^D = {x^D}^\dagger$.

Drazin Inverses for Self-Adjoint Maps

In a †-category, a **self-adjoint map** is a map $x : A \to A$ such that $x = x^\dagger$.

For self-adjoints maps, the Drazin inverses and the dagger Drazin inverse coincide.

Lemma

In a †-category, a self-adjoint map x is †-Drazin if and only if x is Drazin. Moreover, if x is self-adjoint and (†-)Drazin, then its (†-)Drazin inverse is also self-adjoint and $x^D = x^\partial$.

Positive Maps

An important class of self-adjoint maps are the *positive* maps.

In a †-category, a **positive map** is an endomorphism $p : A \to A$ such that there exists a map $f : A \to B$ such that $p = ff^\dagger$.

Corollary

In a †-category, a positive map is †-Drazin if and only if it is Drazin.

Fundamental Theorem of Dagger Drazin Inverses

Now for any map f in a †-category, we have two positive maps associated to it: $ff^\dagger$ and $f^\dagger f$.

Theorem

In a †-category, for a map f, the following are equivalent:

- *f is †-Drazin;*
- *$f^\dagger$ is †-Drazin;*
- *$ff^\dagger$ is Drazin;*
- *$f^\dagger f$ is Drazin.*

Moreover, if f is †-Drazin, then:

- $f^\partial = f^\dagger (ff^\dagger)^D = (f^\dagger f)^D f^\dagger$ *and* $f^{\dagger^\partial} = f(f^\dagger f)^D = (ff^\dagger)^D f$;
- $(ff^\dagger)^D = f^{\dagger^\partial} f^\partial$ *and* $(f^\dagger f)^D = f^\partial f^{\dagger^\partial}$;
- $\mathrm{ind}^\partial(f) = \max\{\mathrm{ind}^D(ff^\dagger), \mathrm{ind}^D(f^\dagger f)\}$.

Corollary

A †-category is †-Drazin if and only if every positive map is Drazin. In particular, every †-category that is Drazin is also †-Drazin.

Example: Matrices

Example

MAT($\mathbb{K}$) is Drazin and also a †-category with respect to T. As such, MAT($\mathbb{K}$) is †-Drazin. So every $\mathbb{K}$-matrix A has a T-Drazin inverse given by:

$$A^{\partial} = A^{\mathsf{T}}(AA^{\mathsf{T}})^{D}$$

Example

If $\mathbb{K}$ is an involutive field, then MAT($\mathbb{K}$) is also a †-category with respect to $*$. As such MAT($\mathbb{K}$) is †-Drazin with respect to $*$, so every $\mathbb{K}$-matrix A has a $*$-Drazin inverse given by:

$$A^{\partial} = A^{*}(AA^{\dagger})^{D}$$

We will revisit this example when $\mathbb{K} = \mathbb{C}$ in a few slides...

Example: Hilbert Spaces

Even in †-categories that are neither Drazin nor †-Drazin, it is sometimes possible to characterize the maps that are †-Drazin.

Example

HILB is not Drazin, nor is it †-Drazin. Nevertheless, we can still characterize the $*$-Drazin maps in HILB, since there is a characterization of when a bounded linear operator is Drazin.

For a bounded linear operator $f : H_1 \to H_2$, let $\mathsf{N}(f)$ denote its null space and $\mathsf{R}(f)$ its range space. A bounded linear operator $g : H \to H$ is said to be has **finite ascent** (resp. **descent**) if there exists a $k \in \mathbb{N}$ such that $\mathsf{N}(g^k) = \mathsf{N}(g^{k+1})$ (resp. $\mathsf{R}(g^k) = \mathsf{R}(g^{k+1})$).

Then a bounded linear operator $g : H \to H$ is Drazin if and only if it has both finite ascent and finite descent.

C. F. King **A note on Drazin inverses.** (1977)

Wei, Y. and Qiao, S. **The representation and approximation of the Drazin inverse of a linear operator in Hilbert space.** (2003)

So a bounded linear operator $f : H_1 \to H_2$ is $*$-Drazin if and only if ff^* (or equivalently f^*f) has both finite ascent and finite descent.

Separating Example

There are †-Drazin categories that are not Drazin.

Example

Let PINJ be the category of sets and partial injections. Then PINJ is a †-category where for a partial injection $f : X \to Y$, $f^\dagger : Y \to X$ is defined as $f^\dagger(y) = x$ if $f(x) = y$ and is undefined otherwise. PINJ is †-Drazin, where $f^\partial = f^\dagger$ (since every map is a partial isometry). However, PINJ is not Drazin!

$$s^D(n) = s^D(s^n(0)) = s^n(s^D(0)) = s^D(0) + n$$

So s^D is completely determined by $s^D(0)$. If $s^D(0)$ is defined, then:

$$k = s^k(0) = s^D(s^{k+1}(0)) = s^D(k+1) = s^D(0) + k + 1$$

This implies that $0 = s^D(0) + 1$ – which is a contradiction since $s^D(0) \in \mathbb{N}$. On the other hand if $s^D(0)$ is undefined, then:

$$s^k(0) = s^D(s^{k+1}(0)) = s^{k+1}(s^D(0))$$

would also be undefined, but this is a contradiction since s^k is total. So the successor function does not have a Drazin inverse, and therefore PINJ is not Drazin.

Dagger Drazin Inverses and Moore-Penrose Inverses

We now turn our attention to looking at another kind of generalized inverse which can be defined in any dagger category: the **Moore-Penrose Inverse**.

Moore-Penrose Inverses

Definition

In a $\dagger$-category, a **Moore-Penrose inverse** (M-P inverse) of a map $f : A \to B$ is a map $f^\circ : B \to A$ such that the following equalities hold:

$$ff^\circ f = f \qquad f^\circ ff^\circ = f^\circ \qquad (ff^\circ)^\dagger = ff^\circ \qquad (f^\circ f)^\dagger = f^\circ f$$

A **Moore-Penrose dagger category** is a dagger category such that every map is M-P invertible.

Lemma

In a dagger category, M-P inverses are unique (if they exist).

R. Cockett & J.-S. P. Lemay **Moore-Penrose Dagger Categories**. (2023)

Dagger Drazin Inverses and Moore-Penrose Inverses

Having a Moore-Penrose inverse is equivalent to being **a** †-Drazin inverse. It is important to note that being a †-Drazin inverse is strictly stronger than being †-Drazin. Indeed, while every †-Drazin inverse is †-Drazin, not every †-Drazin map is itself a †-Drazin inverse.

Theorem

In a †-category, the following are equivalent for a map f:

- *f is Moore-Penrose;*
- *f is a †-Drazin inverse;*
- *f is †-Drazin and $f^{\partial\partial} = f$.*

Therefore, if f is Moore-Penrose then f is †-Drazin where $f^{\partial} = f^{\circ}$ and $\mathrm{ind}^{\partial}(f) \leq 1$, and, moreover, f is the †-Drazin inverse of f°, that is, $f^{\circ\partial} = f$.

Corollary

A †-category is Moore-Penrose if and only if it is †-Drazin where every map is also a †-Drazin inverse.

Example: Complex Matrices

Example

Let $\mathbb{C}$ be the field of complex numbers. MAT($\mathbb{C}$) with the conjugate transpose dagger $*$ is a Moore-Penrose †-category where the Moore-Penrose inverse of a complex matrix can be constructed from its singular value decomposition. Therefore, every complex matrix is $*$-Drazin and its $*$-Drazin inverse is its Moore-Penrose inverse. Of course, we can also say that every complex matrix is the $*$-Drazin inverse of its Moore-Penrose inverse.

Example

FHILB is a Moore-Penrose †-category, and therefore FHILB is also †-Drazin. So every linear operator between finite dimensional Hilbert spaces is $*$-Drazin, where its $*$-Drazin inverse is its Moore-Penrose inverse.

It is important to note that while for any (involutive) field, its category of matrices is always †-Drazin, it may not be Moore-Penrose.

Example

Consider MAT($\mathbb{C}$) this time with simply the transpose T dagger, which we know is †-Drazin, that is, every complex matrix has a T-Drazin inverse. However, MAT($\mathbb{C}$) is not Moore-Penrose with the transpose dagger, in particular, the matrix $\begin{bmatrix} i & 1 \end{bmatrix}$ does not have a Moore-Penrose inverse with respect to the transpose. However $\begin{bmatrix} i & 1 \end{bmatrix}$ does have a T-Drazin inverse, $\begin{bmatrix} i & 1 \end{bmatrix}^{\partial} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathsf{T}}$.

Quick Word on Drazin Opposing Pairs

The last thing I will mention today is that our original attempt at generalizing Drazin inverses in our first paper was using **opposing pairs**.

In an arbitrary category $\mathbb{X}$, an **opposing pair** $(f, g) : A \to B$ is a pair (f, g) consisting of maps of dual type, so $f : A \to B$ and $g : B \to A$. We then went on to define the notion of a Drazin inverse for opposing pairs.

We can recapture this notion using †-Drazin inverses via:

Theorem

Drazin inverses of opposing pairs correspond to the †-Drazin inverses in **cofree** *dagger categories.*

More about Drazin Inverses to come? Future work?

If you found this story interesting and want to discuss/work together on ideas, feel free to reach out to me, Priyaa, or Robin:

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`https://sites.google.com/view/jspl-personal-webpage/`

HOPE YOU ENJOYED MY TALK!

THANK YOU FOR LISTENING!

MERCI!