General pointfree theories of T_0 spaces

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Table of contents

The duality

The classical pointfree versions of topological spaces are *frames*. A frame is a complete lattice L where the distributivity law

$$(\bigvee_i x_i) \wedge y = \bigvee_i (x_i \wedge y)$$

holds. Frames form the category **Frm** when equipped with maps which preserve arbitrary joins and finite meets.

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holds. Frames form the category **Frm** when equipped with maps which preserve arbitrary joins and finite meets. We have an adjunction

$$\mathbf{Frm}^{op} \underset{\Omega}{\overset{\mathrm{pt}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{Top}$$

Sublocales

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- S is closed under all meets;
- If $s \in S$ and $x \in L$ then $x \to s \in S$.

We will mention some important facts about sublocales.

Theorem

The ordered collection $\mathsf{S}(L)$ of sublocales of a frame L is a coframe. Meets are set-theoretical intersections.

Open and closed sublocales

Every element $a \in L$ determines an *open* sublocale $\mathfrak{o}(a) \subseteq L$ and a *closed* one $\mathfrak{c}(a) \subseteq L$. We will only

- The collection $\mathsf{S}_{\mathfrak{o}}(L) \subseteq \mathsf{S}(L)$ of intersections of closed sublocales is a subcoframe of $\mathsf{S}(L)$.
- The collection $\mathsf{S}_{\mathfrak{c}}(L) \subseteq \mathsf{S}(L)$ of joins of closed sublocales is a subframe of $\mathsf{S}(L)$.
- The collection $\mathsf{S}_{b}(L) \subseteq \mathsf{S}(L)$ of joins of complemented sublocales is a subcolocale, in particular it is the Booleanization of $\mathsf{S}(L)$.

Raney extensions

We now introduce *Raney extensions*. Today we give a definition equivalent to the original one. A *Raney extension* is a pair $(L, \mathcal{F})$ such that L is a frame and $\mathcal{F} \subseteq \mathsf{S}_{\mathfrak{o}}(L)$ is a subcolocale. A morphism $f : (L, \mathcal{F}) \to (M, \mathcal{G})$ between Raney extensions is a frame map $f : L \to M$ compatible with $\mathcal{F}$ and $\mathcal{G}$ in a precise sense beyond the scope of this talk.

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Example

- For a topological space, let $\mathcal{U}(X)$ be the ordered collection of intersections of open sets, called the *saturated* sets. The pair $(\Omega(X), \mathcal{U}(X))$ is a Raney extension.
- The pair $(L, \mathsf{S}_{\mathfrak{o}}(L))$ is obviously a Raney extension for every frame L.
- The pair $(L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$ is a Raney extension for every frame L.

Raney extensions

We regard Raney extensions as pointfree T_0 spaces in light of the following result.

Theorem

There is a dual adjunction $\Omega_R : \mathbf{Top} \leftrightarrows \mathbf{Raney}^{op} : \mathrm{pt}_R$*, extending* $\mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathrm{pt}$*, such that the fixpoints in* **Top** *are the* T_0 *spaces.*

Raney extensions

The Raney extensions $(L, \mathsf{S}_{\mathfrak{o}}(L))$ and $(L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$ are especially significant.

Theorem

For every Raney extension (L, C) there are surjections

$$(L, \mathsf{S}_{\mathfrak{o}}(L)) \xrightarrow{\Sigma_{(L,C)}} (L, C) \xrightarrow{\tau_{(L,C)}} (L, \mathsf{S}_{\mathfrak{c}}(L)^{op}).$$

Up to homeomorphism, the dualizations of these surjections are subspace inclusions:

$$\mathsf{pt}_D(L) \xrightarrow{\mathsf{pt}_R(\tau)} \mathsf{pt}_R(L, C) \xrightarrow{\mathsf{pt}_R(\Sigma)} \mathsf{pt}(L).$$

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In fact, the assignment $L \mapsto (L, \mathsf{S}_{\mathfrak{o}}(L))$ determines a functor $\mathsf{S}_{\mathfrak{o}} : \mathbf{Frm} \to \mathbf{Raney}$. This is left adjoint to the forgetful functor $\pi_1 : \mathbf{Raney} \to \mathbf{Frm}$. The assignment $L \mapsto (L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$ is not functorial. We call $\mathbf{Frm}_{\mathcal{E}}$ the wide subcategory of $\mathbf{Frm}$ whose morphisms are the *exact* maps, the ones who do lift.

Strictly zero-dimensional biframes

Another approach is that of strictly zero-dimensional biframes. These were studied in detail by Manuell in *Strictly zero-dimensional biframes and a characterization of congruence frames* (2018). A *stricly zero-dimensional biframe* may be identified with a pair $(L, \mathcal{D})$ where L is a frame, and $\mathcal{D}$ is a dense subcolocale of $\mathsf{S}(L)$. A morphism $f : (L, \mathcal{D}) \to (M, \mathcal{E})$ between stricly zero-dimensional biframes is a frame map $f : L \to M$ compatible with $\mathcal{D}$ and $\mathcal{E}$.

Example

- For a space X, the pair $(\Omega(X), \mathsf{Sk}(X))$, where $\mathsf{Sk}(X)$ are the opens in the Skula topology, is a strictly zero-dimensional biframe.
- The pair $(L, \mathsf{S}(L))$ is obviously a strictly zero-dimensional biframe for every frame L.
- The pair $(L, \mathsf{S}_b(L))$ is a strictly zero-dimensional biframe for every frame L.

Strictly zero-dimensional biframes

We regard strictly zero-dimensional biframes as pointfree T_0 spaces in light of the following result.

Theorem

There is a dual adjunction $\mathrm{Sk} : \mathbf{Top} \leftrightarrows \mathbf{SZD}^{op} : \mathrm{pt}_B$*, extending* $\mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathrm{pt}$*, such that the fixpoints in* $\mathbf{Top}$ *are the* T_0 *spaces.*

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There is a dual adjunction $\mathsf{Sk} : \mathbf{Top} \leftrightarrows \mathbf{SZD}^{op} : \mathsf{pt}_B$, *extending* $\mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathsf{pt}$, *such that the fixpoints in* **Top** *are the* T_0 *spaces.*

The functor Sk sends a space X to the strictly zero-dimensional biframe $(\Omega(X), \mathsf{Sk}(X))$

Strictly zero-dimensional biframes

For a frame L, we have the strictly zero-dimensional biframes $(L, \mathsf{S}(L))$ and $(L, \mathsf{S}_b(L))$. The following results are gathered from Manuell, 2018.

Theorem

For every strictly zero-dimensional biframe $(L, \mathcal{D})$ *there are surjections*

$$(L, \mathsf{S}(L)) \xrightarrow{\Delta_{(L,C)}} (L, \mathcal{D}) \xrightarrow{\iota_{(L,C)}} (L, \mathsf{S}_b(L)).$$

Up to homeomorphism, the dualizations of these surjections are subspace inclusions:

$$\mathrm{pt}_D(L) \xrightarrow{\mathrm{pt}_B(\iota)} \mathrm{pt}_B(L, \mathcal{D}) \xrightarrow{\mathrm{pt}_B(\Delta)} \mathrm{pt}(L).$$

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Once again, the assignment $L \mapsto (L, \mathsf{S}(L))$ is functorial and is left adjoint to the forgetful functor to **Frm**. But the assignment $L \mapsto (L, \mathsf{S}_b(L))$ is not. We call $\mathbf{Frm}_{\mathcal{B}}$ the subcategory of **Frm** whose morphisms lift to this.

Dualizing objects

We follow definitions from Tholen and Porst, *Concrete dualities*, 1991. Suppose there are categories $\mathcal{A}$ and $\mathcal{B}$ with underlying set functors $U : \mathcal{A} \to \mathbf{Set}$ and $V : \mathcal{B} \to \mathbf{Set}$.

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If this holds, for $X \in \mathcal{A}$, and $x \in UX$, there is an evaluation map

$$\varepsilon_{(X,x)} : \mathcal{A}(X, A) \to U(A),$$

defined as $\varepsilon_{(X,x)}(s) = Us(x)$ for all $s \in \mathcal{A}(X, A)$. Similarly, there is another evaluation map

$$\eta_{(Y,y)} : \mathcal{B}(Y, B) \to VB,$$

defined as $\eta_{(Y,y)}(t) = Vt(y)$.

Dualizing objects

A triple (A, B, τ) with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $\tau : UA \cong VB$ a bijection with inverse σ, is called a *dualizing object* if the following two properties hold.

1. For all $X \in \mathcal{A}$, the family $\tau\varepsilon_{(X,a)} : \mathcal{A}(X, A) \to VB$, for $a \in UA$, admits an initial lifting $\tau\varepsilon_{(A,a)} : TA \to B$.
2. For all $Y \in \mathcal{B}$, the family $\sigma\eta_{(Y,b)} : \mathcal{B}(Y, B) \to UA$, for $b \in VY$, admits an initial lifting $\sigma\eta_{(Y,b)} : SY \to A$.

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Theorem

For every dualizing object (A, B, τ), there is a dual adjunction $T : \mathcal{A} \leftrightarrows \mathcal{B} : S$ strictly represented by A and B.

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Theorem

For every dualizing object (A, B, τ), there is a dual adjunction $T : \mathcal{A} \leftrightarrows \mathcal{B} : S$ strictly represented by A and B. In particular:

1. $US = \mathcal{B}(-, B)$ *and* $VT = \mathcal{A}(-, A)$.
2. *For* $X \in \mathcal{A}$ *and* $x \in UX$, *for each* $x \in UX$ *the unit is defined as* $x \mapsto \varepsilon_{(X,x)}$, *where the latter is the initial lifting* $\varepsilon_{(X,x)} : TX \to B$.

General theories of T_0 spaces

A *pointfree category of T_0 spaces* consists of

- A functor $\mathcal{O} : \mathcal{C} \to \mathbf{Frm}$ from some category $\mathcal{C}$;
- An object $2_{\mathcal{C}} \in \mathcal{C}$;

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such that:

- $\mathcal{O}$ is faithful and essentially surjective;
- The pair $(2_{\mathcal{C}}, \mathbb{S})$ is a dualizing object;
- The fixpoints in **Top** of the resulting adjunction $(-, \mathbb{S}) : \mathbf{Top} \leftrightarrows \mathcal{C} : (-, 2_{\mathcal{C}})$ are the T_0 spaces.

General theories of T_0 spaces

We call $\Omega_{\mathcal{C}}$: and $\mathrm{pt}_{\mathcal{C}}$: the two functors arising from the pair $(2_{\mathcal{C}}, \mathbb{S})$. We note that the category $\mathcal{C}$ is understood to have $V \circ \mathcal{O}$ as the underlying set functor, where $V : \mathbf{Frm} \to \mathbf{Set}$ is that functor for frames. Then $\mathbf{Top}(X, \mathbb{S})$ is just the underlying set of the frame $\mathcal{O}(\Omega_{\mathcal{C}}(X))$.

Lemma

If $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ is a pointfree category of T_0 spaces, the functors $\mathcal{O} \circ \Omega_{\mathcal{C}}$ and Ω are naturally isomorphic.

Proof.

(Sketch). We prove that the two underlying sets are the same. Both $\Omega_{\mathcal{C}} \dashv \mathrm{pt}_{\mathcal{C}}$ and $\Omega \dashv \mathrm{pt}$ have $\mathbb{S}$ as a dualizing object, and so $V\mathcal{O}\Omega_{\mathcal{C}}(X) \cong \mathbf{Top}(X, \mathbb{S}) \cong V\Omega X$. □

We call $\zeta : \mathcal{O}\Omega_{\mathcal{C}} \to \Omega$ the natural isomorphism given by the lemma above.

General theories of T_0 spaces

Proposition

If $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ is a pointfree theory of T_0 spaces there is a diagram as follows in which the right adjoints commute up to natural isomorphism.

$$
\begin{array}{ccc}
\mathcal{C} & \underset{\Omega_{\mathcal{C}}}{\overset{\mathrm{pt}_{\mathcal{C}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathbf{Top}^{\mathrm{op}} \\
\Big\downarrow{\scriptstyle \mathcal{O}} & & \Big\| \\
\mathbf{Frm} & \underset{\Omega}{\overset{\mathrm{pt}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathbf{Top}^{\mathrm{op}}
\end{array}
$$

In other words, $\mathcal{O}$ determines a lax map of adjunctions.

General theories of T_0 spaces

Let η and $\eta^{\mathcal{C}}$ be the units in **Frm** and in $\mathcal{C}$, respectively. Let σ, and $\sigma^{\mathcal{C}}$ be the corresponding counits. Recall that by our main definition we know the following.

Fact

The counit $\sigma^{\mathcal{C}}_X$ is a homeomorphism for every $X \in$ **Top**.

General theories of T_0 spaces

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Fact

*The counit $\sigma^{\mathcal{C}}_X$ is a homeomorphism for every $X \in$ **Top**.*

We can also prove an important fact about the unit $\eta_{\mathcal{C}}$.

Lemma

Let $\mathcal{C}$ be a category of T_0 spaces. For $C \in \mathcal{C}$, $\zeta_{\mathrm{pt}_{\mathcal{C}} C} \circ \mathcal{O}\eta_C : \mathcal{O}C \to \Omega \mathrm{pt}_{\mathcal{C}} C$ satisfies

$$\mathcal{O}\eta_C(a) = \{f \in \mathrm{pt}_{\mathcal{C}}(C) \mid \mathcal{O}f(a) = 1\}.$$

In particular, this is a frame surjection.

General theories of T_0 spaces

Mate theory guarantees the existence of a natural transformation $\xi : \mathsf{pt}_{\mathcal{C}} \to \mathsf{pt} \circ \mathcal{O}$. This is the composition:

$$\mathsf{pt}_{\mathcal{C}} C \xrightarrow{\sigma_{\mathsf{pt}_{\mathcal{C}} C}} \mathsf{pt}\Omega\mathsf{pt}_{\mathcal{C}} C \xrightarrow{\mathsf{pt}\zeta_{\mathsf{pt}_{\mathcal{C}} C}} \mathsf{pt}\mathcal{O}\Omega_{\mathcal{C}}\mathsf{pt}_{\mathcal{C}} C \xrightarrow{\mathsf{pt}\mathcal{O}(\eta^{\mathcal{C}}_{C})} \mathsf{pt}\mathcal{O} C.$$

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Proposition

The map $\xi_C : \mathrm{pt}_{\mathcal{C}}(C) \to \mathrm{pt}(\mathcal{O}(C))$ is a subspace embedding for all $C \in \mathcal{C}$.

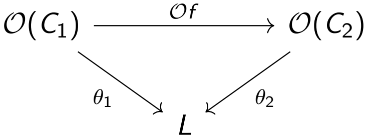
Proof.

We look at the diagram above. The sobrification $\sigma_{\mathrm{pt}_{\mathcal{C}} C}$ is a subspace embedding as $\mathrm{pt}_{\mathcal{C}} C$ is a T_0 space. The second morphism is an isomorphism, as $\zeta_{\mathrm{pt}_{\mathcal{C}} C}$ is. The third is a subspace embedding, as we have shown that $\mathcal{O}(\eta^{\mathcal{C}}_C)$ is a frame surjection. □

Initial and terminal objects of fibers

For a pointfree theory of T_0 spaces $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$, we define the *fiber* $\mathcal{O}^{-1}(L)$ of a frame L to be the category:

- Whose objects are pairs (C, θ) where $C \in \mathcal{C}$ and $\theta : \mathcal{O}C \cong L$ is an isomorphism;
- Whose morphisms $f : (C_1, \theta_1) \to (C_2, \theta_2)$ are maps $f : C_1 \to C_2$ in $\mathcal{C}$ such that the following commutes in **Frm**.



Initial and terminal objects of fibers

We call $\mathcal{C}_I$ be the full subcategory of $\mathcal{C}$ determined by the initial objects of the fibers, and $\mathcal{C}_T$ that of the terminal ones. Consider the composite functor below, where $I_{\mathcal{I}}$ is subcategory inclusion.

$$\mathcal{C}_{\mathcal{I}} \xhookrightarrow{I_{\mathcal{I}}} \mathcal{C} \xrightarrow{\mathcal{O}} \mathbf{Frm}$$

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The functor is faithful. It is essentially surjective iff every frame has an initial object in its fiber. In general, this will not be full. We call $\mathbf{Frm}_{\mathcal{I}}$ the essential image of this functor.

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$$\mathcal{C}_{\mathcal{T}} \xhookrightarrow{I_{\mathcal{T}}} \mathcal{C} \xrightarrow{\mathcal{O}} \mathbf{Frm},$$

the analogous construction for terminal objects.

Initial and terminal objects of fibers

We say that a pointfree theory $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ of T_0 spaces is *bounded* if every fiber has both an initial and a terminal object.

Example

SZD and **Raney** are both bounded. As we have seen:

- For a Raney extension $(L, \mathcal{F})$ the initial object of its fiber is $(L, \mathsf{S}_{\mathfrak{o}}(L))$, and the terminal one is $(L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$.
- For a strictly zero-dimensional biframe $(L, \mathcal{D})$ the initial object of its fiber is $(L, \mathsf{S}(L))$, and the terminal one is $(L, \mathsf{S}_b(L))$.

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Proposition

If $C \in \mathcal{C}$ is such that its fiber $\mathcal{O}^{-1}(\mathcal{O}(C))$ has both an initial object I and a terminal object T, there are subspace embeddings

$$\mathrm{pt}_{\mathcal{C}}(T) \subseteq \mathrm{pt}_{\mathcal{C}}(C) \subseteq \mathrm{pt}_{\mathcal{C}}(I).$$

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Boundedness and reasonableness

We say that $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ is *reasonable* if in $\mathcal{O}^{-1}(2)$ the object $2_{\mathcal{C}}$ is both initial and terminal. We now define two new spectrum functors. For a frame L we define $\mathrm{pt}_{\mathcal{I}}(L)$ as the set of all morphisms $f : L \to 2$ in $\mathbf{Frm}_{\mathcal{I}}$. We topologize it via the usual Stone map. We define $\mathrm{pt}_{\mathcal{T}}$ similarly. Recall the subspace embedding $\xi_C : \mathrm{pt}_{\mathcal{C}} C \to \mathrm{pt}\mathcal{O}C$. We call ξ^I and ξ^T the co-restrictions to $\mathrm{pt}_{\mathcal{I}}\mathcal{O}C$ and $\mathrm{pt}_{\mathcal{T}}\mathcal{O}C$, respectively.

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Proposition

In a reasonable pointfree theory of T_0 spaces $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$, and $L \in \mathbf{Frm}$:

1. *If $I \in \mathcal{O}^{-1}(L)$ is an initial object, $\xi^I : \mathrm{pt}_{\mathcal{C}}(I) \to \mathrm{pt}_{\mathcal{I}}(L)$ is a homeomorphism.*
2. *If $T \in \mathcal{O}^{-1}(L)$ is a terminal object, $\xi^T : \mathrm{pt}_{\mathcal{C}}(T) \to \mathrm{pt}_{\mathcal{T}}(L)$ is a homeomorphism.*

Boundedness and reasonableness

Theorem

If $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ *is bounded and reasonable, all* $C \in \mathcal{C}$ *there are subspace embeddings*

$$\mathrm{pt}_{\mathcal{T}}(\mathcal{O}(C)) \hookrightarrow \mathrm{pt}_{\mathcal{C}}(C) \hookrightarrow \mathrm{pt}_{\mathcal{I}}(\mathcal{O}(C)).$$

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Example

Raney and **SZD** are both bounded and reasonable.

- For **Raney**, $\mathbf{Frm}_{\mathcal{T}}$ is $\mathbf{Frm}_{\mathcal{E}}$. The subspace inclusion above is $\mathrm{pt}_D(L) \subseteq \mathrm{pt}_R(L, \mathcal{F}) \subseteq \mathrm{pt}(L)$.

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Example

Raney and **SZD** are both bounded and reasonable.

- For **Raney**, $\mathbf{Frm}_{\mathcal{T}}$ is $\mathbf{Frm}_{\mathcal{E}}$. The subspace inclusion above is $\mathrm{pt}_D(L) \subseteq \mathrm{pt}_R(L, \mathcal{F}) \subseteq \mathrm{pt}(L)$.
- For **SZD**, $\mathbf{Frm}_{\mathcal{T}}$ is $\mathbf{Frm}_{\mathcal{B}}$. The subspace inclusion above is $\mathrm{pt}_D(L) \subseteq \mathrm{pt}_B(L, \mathcal{D}) \subseteq \mathrm{pt}(L)$.

Duality for terminal objects

We say that $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ is *spatially terminal* if whenever T is terminal in its fiber so is its spatialization $\Omega_{\mathcal{C}}\mathrm{pt}_{\mathcal{C}}T$.

Example

Both **Raney** and **SZD** are terminally spatial.

We begin constructing the duality. We call $\mathbf{Top}_{\mathcal{T}}$ the essential image of the functor $\mathrm{pt}_{\mathcal{T}} : \mathbf{Frm}^{op} \to \mathbf{Top}$.

Proposition

If $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ is spatially terminal, then $\mathbf{Top}_{\mathcal{T}}$ is a full subcategory of $\mathbf{Top}$.

Duality for terminal objects

Lemma

Suppose that $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ *is bounded, reasonable, and spatially terminal. For every frame* L*, the map* $\varphi^T : L \to \Omega\mathrm{pt}_{\mathcal{T}} L$ *defined as*

$$\eta^T(a) = \{f \in \mathrm{pt}_{\mathcal{T}}(L) \mid f(a) = 1\}$$

is in $\mathbf{Frm}_{\mathcal{T}}$*.*

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Lemma

Suppose that $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ *is bounded, reasonable, and spatially terminal. For a space* $X \in \mathbf{Top}_{\mathcal{T}}$*, the characteristic map* $\Omega X \to 2$ *is in* $\mathbf{Frm}_{\mathcal{T}}$ *for every point* $x \in X$*. So, there is a well-defined map* $\sigma^T : X \to \mathrm{pt}_{\mathcal{T}} \Omega X$*.*

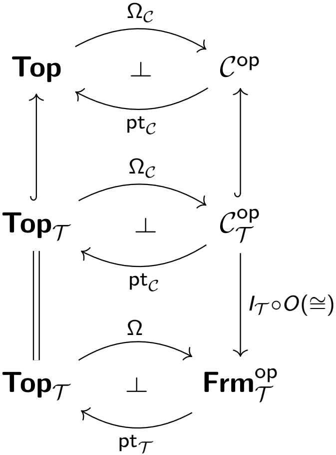
Duality for terminal objects

We have reached the main theorem.

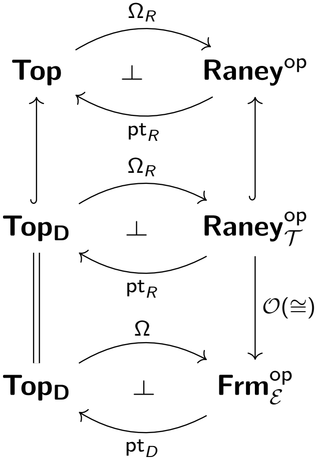
Theorem

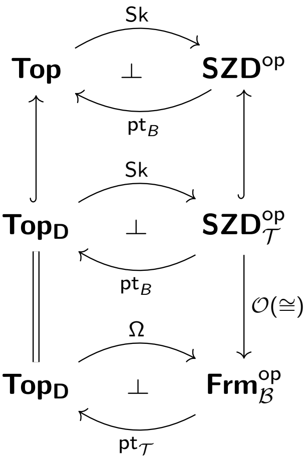
Suppose that $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, 2_{\mathcal{C}})$ *is bounded, reasonable, and spatially terminal. There is an adjunction* $\Omega : \mathbf{Top}_{\mathcal{T}} \leftrightarrows \mathbf{Frm}_{\mathcal{T}}^{op} : \mathrm{pt}_{\mathcal{T}}$ *with* $\Omega \dashv \mathrm{pt}_{\mathcal{T}}$ *where all elements of* $\mathbf{Top}_{\mathcal{T}}$ *are fixpoints.*

Duality for terminal objects: the big picture



Duality for terminal objects: concrete examples





Duality for initial objects

We may want to look at the case where $\mathcal{C}$ has free objects. In this case, obtaining a similar duality for initial objects is very simple.

Proposition

If $\mathcal{O} : \mathcal{C} \to \mathbf{Frm}$ has a left adjoint $\mathcal{F} : \mathbf{Frm} \to \mathcal{C}$, then for every frame L its fiber has $\mathcal{F}L$ as its initial object.

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Example

In both **Raney** and **SZD** the forgetful functor to **Frm** has a left adjoint. For a frame L, the initial object of the fiber in **Raney** is $(L, \mathsf{S}_{\mathfrak{o}}(L))$, and the initial object of the fiber in **SZD** is $(L, \mathsf{S}(L))$.

Duality for initial objects

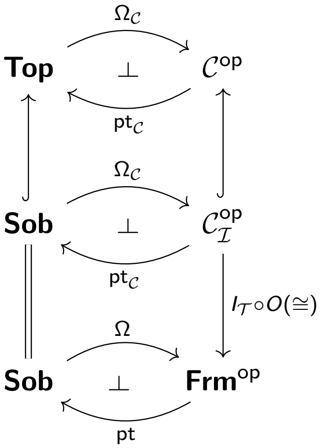
The duality for initial objects, under relatively weak assumptions, collapses to the usual duality for sober spaces.

Theorem

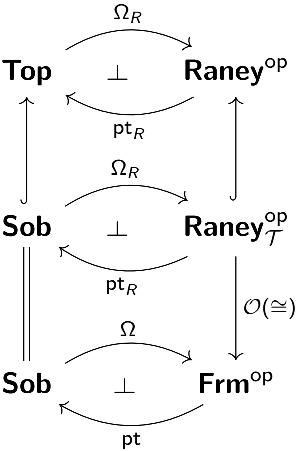
If $\mathcal{O} : \mathcal{C} \to \mathbf{Frm}$ *has a left adjoint, and is reasonable, then*

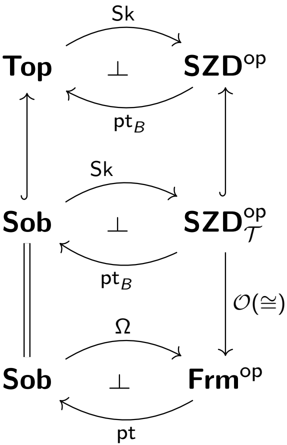
1. $\mathbf{Frm}_{\mathcal{I}} = \mathbf{Frm}$;
2. $\mathrm{pt}_{\mathcal{I}} = \mathrm{pt}$;
3. $\mathbf{Top}_{\mathcal{I}} = \mathbf{Sob}$.

Duality for initial objects: the big picture



Duality for initial objects: concrete examples





Open questions

Then, for both **Raney** and **SZD**, even if $\mathbf{Frm}_{\mathcal{E}}$ and $\mathbf{Frm}_{\mathcal{B}}$ are different, the spectra induced by them are both pt_D. Equivalently, any terminal element $T \in \mathcal{C}$ of some fiber is always such that $\mathrm{pt}_{\mathcal{C}}(T) \cong \mathrm{pt}_D(\mathcal{O}T)$. The T_D axiom, and a duality for it, in both cases arises naturally when restricting to terminal objects.

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Is this the case for all pointfree theories of T_0 spaces? If not, what are some natural assumptions on a pointfree theory $\mathcal{O} : \mathcal{C} \to \mathbf{Frm}$ of T_0 spaces that imply this? Can we define a useful such $\mathcal{O} : \mathcal{C} \to \mathbf{Frm}$ where the terminal objects of the fibers are some other class of spaces, e.g. the T_1 spaces?

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Another possible direction is to define a notion of morphism between pointfree theories of T_0 spaces, and check how close **Raney** or **SZD** are to being universal in some sense in this category.