The invariant ring of pairs of matrices

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Joint work with Farkhod Eshmatov and Rustam Turdibaev

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$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad XX = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad XY = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

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Is $\mathrm{Tr}(X^2Y)$ related to $\mathrm{Tr}(X)$, $\mathrm{Tr}(X^2)$, $\mathrm{Tr}(Y)$ and $\mathrm{Tr}(XY)$?

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So, taking traces both sides:

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And therefore,

$$\operatorname{Tr}(X^2Y) = \operatorname{Tr}(X)\operatorname{Tr}(XY) - \frac{1}{2}\operatorname{Tr}(X)^2\operatorname{Tr}(Y) + \frac{1}{2}\operatorname{Tr}(X^2)\operatorname{Tr}(Y)$$

Motivating question

It is a classical result that for any word we can form with X and Y, its trace can be written uniquely as a polynomial in the five variables:

$$\mathrm{Tr}(X) \quad \mathrm{Tr}(X^2) \quad \mathrm{Tr}(XY) \quad \mathrm{Tr}(Y^2) \quad \mathrm{Tr}(Y)$$

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It was proved in 2006 that we need the following 11 variables:

$$
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\mathrm{Tr}(X^2) & \mathrm{Tr}(XY) & \mathrm{Tr}(Y^2) & \\
\mathrm{Tr}(X^3) & \mathrm{Tr}(X^2Y) & \mathrm{Tr}(XY^2) & \mathrm{Tr}(Y^3) \\
\mathrm{Tr}(X^2Y^2) & & & \\
\mathrm{Tr}(X^2Y^2XY) & & &
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But there is a relation among them in degree 12.

Case $n = 4$

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It is the main purpose of this talk.

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A very classical problem in invariant theory, asks to describe the ring of invariants:

$$C_{nd} := \mathbb{C}[\mathcal{M}_n^d]^{\mathrm{GL}_n}$$

Invariant theory

A classical result says that C_{nd} has the following properties:

- It is generated uniquely by trace functions.

C. Procesi
The invariant theory of $n \times n$ matrices
Advances in Mathematics 19 (3), 306–381, (1976).

Ju. P. Razmyslov,
Identities with trace in full matrix algebras over a field of characteristic zero,
Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya, 8 (4) 723–756, (1974).

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Since it is Cohen-Macaulay, we can try to describe it as a free module over $\mathbb{C}[a_1, \ldots, a_r]$ for some homogeneous system of parameters $a_1, \ldots, a_r$ (Hironaka decomposition).

Trace functions

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2) They are *cyclic*, i.e.

$$\mathrm{Tr}(XXYXY) = \mathrm{Tr}(YXXYX) = \mathrm{Tr}(XYXXY) = \mathrm{Tr}(YXYXX)$$

Traceless matrices

Also, it is interesting to see that we can convert any matrix in a traceless one by doing the following operation:

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$$\begin{aligned}
\operatorname{Tr}(A) &= \operatorname{Tr}(X - \frac{1}{n}\operatorname{Tr}(X)I_n) \\
&= \operatorname{Tr}(X) - \operatorname{Tr}(\frac{1}{n}\operatorname{Tr}(X)I_n) \\
&= \operatorname{Tr}(X) - \frac{1}{n}\operatorname{Tr}(X)\operatorname{Tr}(I_n) \\
&= \operatorname{Tr}(X) - \frac{1}{n}\operatorname{Tr}(X)n \\
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As a convention we will use X and Y to talk about any matrix,
and by A and B to talk about their correspondent traceless version.

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where

$$\begin{aligned} a_1 &= \mathrm{Tr}(X) \\ a_2 &= \mathrm{Tr}(Y) \\ a_3 &= \mathrm{Tr}(A^2) \\ a_4 &= \mathrm{Tr}(AB) \\ a_5 &= \mathrm{Tr}(B^2) \end{aligned}$$

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Ya. Dubnov
Complete system of invariants of two affinors in centro-affine space of two or three dimensions
Tr. Semin. Vektorn. Tenzorn. Anal. 5, 250–270, (1941).

C_{32}

Teranishi found that we need 11 generators to describe C_{32}.

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The ring of invariants of matrices,
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However, Nakamoto showed that the square of $\mathrm{Tr}(X^2Y^2XY)$ can be written as a polynomial in terms of the other 10 generators. Therefore it is not a free algebra.

K. Nakamoto,
The structure of the invariant ring of two matrices of degree 3,
Journal of Pure and Applied Algebra 166, 125–148, (2002).

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However, Nakamoto showed that the square of $\mathrm{Tr}(X^2Y^2XY)$ can be written as a polynomial in terms of the other 10 generators. Therefore it is not a free algebra.
Aslaksen, Drensky and Sadikova, choosing traceless generators, found a much simpler description.

$$C_{32} \cong \frac{\mathbb{C}[a_1, \ldots, a_{11}]}{r}$$

$$a_1 = \mathrm{Tr}(X) \quad a_2 = \mathrm{Tr}(Y)$$
$$a_3 = \mathrm{Tr}(A^2) \quad a_4 = \mathrm{Tr}(AB) \quad a_5 = \mathrm{Tr}(B^2)$$
$$a_6 = \mathrm{Tr}(A^3) \quad a_7 = \mathrm{Tr}(A^2B) \quad a_8 = \mathrm{Tr}(AB^2) \quad a_9 = \mathrm{Tr}(B^3)$$
$$a_{10} = \mathrm{Tr}(A^2B^2) - \mathrm{Tr}(ABAB)$$
$$a_{11} = \mathrm{Tr}(A^2B^2AB) - \mathrm{Tr}(B^2A^2BA)$$

H. Aslaksen, V. Drensky, L. Sadikova,
Defining relations of invariants of two 3×3 matrices,
Journal of Algebra 298, 41–57, (2006).

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In fact, while working with traceless matrices the relation is free of a_1 and a_2, so it is possible to write like this:

$$C_{32} \cong \frac{\mathbb{C}[a_1, \dots, a_{11}]}{r} \cong \mathbb{C}[a_1, a_2] \otimes \frac{\mathbb{C}[a_3, \dots, a_{11}]}{r}$$

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In addition, the Hironaka decomposition is:

$$C_{32} \cong \mathbb{C}[a_1, \dots, a_{10}] \oplus a_{11}\mathbb{C}[a_1, \dots, a_{10}]$$

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It is generated by 32 elements. The highest being of degree 10.

V. Drensky, L. Sadikova,
Generators of invariants of two 4×4 matrices,
Comptes rendus de l'Academie bulgare des Sciences 59 (5) 477–484, (2006).

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The 17 primary invariants were found.

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Some independent relations were found in degrees 12, 13 and 14.

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Defining relations of low degree of invariants of two 4×4 matrices,
International Journal of Algebra and Computation 19 (1), 107–127, (2009).

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The Hilbert series was computed.

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The ring of invariants of matrices,
Nagoya Mathematical Journal 104, 149–161, (1986).

A. Berele, J.R. Stembridge,
Denominators for the Poincaré series of invariants of small matrices,
Israel Journal of Mathematics 114, 157–175, (1999).

C_{n2}, with higher n

What is known about C_{n2} when $n > 4$?

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Poincaré series of some pure and mixed trace algebras of two generic matrices,
Journal of Algebra 309 (2), 654–671, (2007).

C_{n2}, with higher n

What is known about C_{n2} when $n > 4$?

It is known that C_{52} is generated by 173 polynomials.

Only the Hilbert series is known for C_{62}.

D. Ž. Đoković,
Poincaré series of some pure and mixed trace algebras of two generic matrices,
Journal of Algebra 309 (2), 654–671, (2007).

C_{2d} and C_{33}

When considering d-tuples of 2×2 matrices, all generators and relations are found and described.

V. Drensky,
Defining relations for the algebra of invariants of 2×2 matrices,
Algebras and Representation Theory 6, 193–214, (2003).

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V. Drensky,
Defining relations for the algebra of invariants of 2×2 matrices,
Algebras and Representation Theory 6, 193–214, (2003).

When considering triples of 3×3 matrices, all generators and relations are found and described.

T. Hoge,
A presentation of the trace algebra of three 3×3 matrices,
Journal of Algebra 358, 257–268, (2012).

A new method

The usual methods to compute a description of C_{nd} usually involve representation theory and combinatorics.

We will present here the completely new method we used to fully solve the problem of C_{42}.

Kontsevich bracket

Consider the free non-commutative algebra generated by two generators $\mathbb{C}\langle x, y\rangle$. This is, the vector space generated by words in the letters x and y.

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There is a well defined bracket

$$\{u_1 \cdots u_p,\ v_1 \cdots v_q\} = \sum_{i=1}^{p} \sum_{j=1}^{q} \omega(u_i, v_j) v_1 \cdots v_{j-1} u_{i+1} \cdots u_p u_1 \cdots u_{i-1} v_{j+1} \cdots v_q,$$

where elements u_i and v_j are either x or y and

$$\omega(x, y) = -\omega(y, x) = 1, \ \ \omega(x, x) = \omega(y, y) = 0.$$

Necklace Lie algebra

It can be proven that it defines a Leibniz algebra structure, i.e.,

$$\big\{a, \{b, c\}\big\} = \big\{\{a, b\}, c\big\} + \big\{b, \{a, c\}\big\}$$

for any words a, b, c.

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Moreover, if we quotient $\mathbb{C}\langle x, y\rangle$ by cyclic words, i.e.,

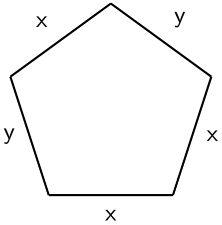
$$xxyxy = yxxyx = xyxxy = yxyxx$$

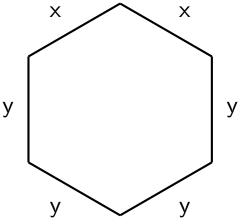
it forms a Lie algebra. It is called Necklace Lie algebra and it is denoted by $\mathcal{L}$.

V. Ginzburg,
Non-commutative symplectic geometry, quiver varieties, and operads,
Mathematical Research Letters 8, 377–400, (2001).

Why Necklace Lie algebra?

$$\{u_1 \cdots u_p,\ v_1 \cdots v_q\} = \sum_{i=1}^{p} \sum_{j=1}^{q} \omega(u_i, v_j) v_1 \cdots v_{j-1} u_{i+1} \cdots u_p u_1 \cdots u_{i-1} v_{j+1} \cdots v_q,$$





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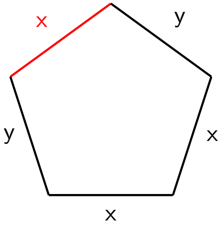

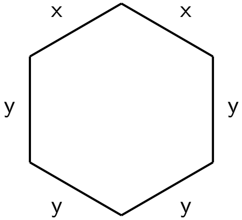

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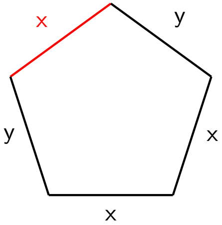

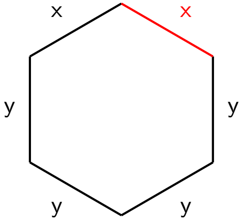

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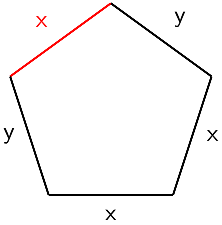
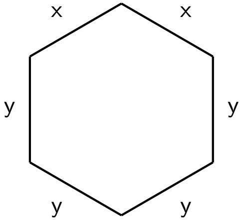
$$\{u_1 \cdots u_p,\ v_1 \cdots v_q\} = \sum_{i=1}^{p} \sum_{j=1}^{q} \omega(u_i, v_j) v_1 \cdots v_{j-1} u_{i+1} \cdots u_p u_1 \cdots u_{i-1} v_{j+1} \cdots v_q,$$





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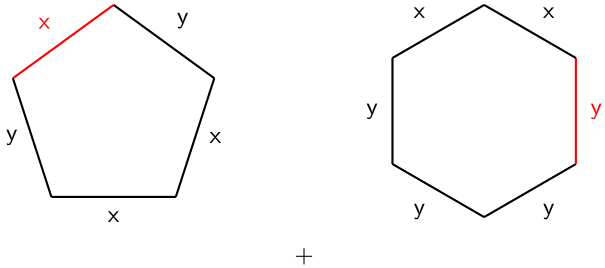

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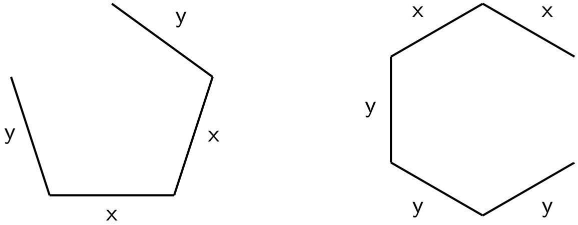

$+yxxy\,yyyxx$

Necklaces and traces

There is a canonical map from the Necklace Lie algebra, to C_{n2}, that sends any word to the trace function of that word:

$$tr\colon \mathcal{L} \to \mathbb{C}[\mathcal{M}_n^2]^{\mathrm{GL}_n} = C_{n2}$$
$$x^{k_1} y^{l_1} \cdots x^{k_m} y^{l_m} \mapsto \big((X, Y) \mapsto \mathrm{Tr}(X^{k_1} Y^{l_1} \cdots X^{k_m} Y^{l_m})\big)$$

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Moreover, it induces a morphism from the tensor algebra over $\mathcal{L}$, inducing a Poisson structure on C_{n2}.

Poisson structure

A Poisson algebra is a vector space equipped with two multiplications:

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Therefore, we have a Poisson algebra structure on the space of traces C_{n2}.

For instance,

$$\begin{aligned}\{\operatorname{Tr}(XY), \operatorname{Tr}(Y)\operatorname{Tr}(XX)\} &= \operatorname{Tr}(Y)\{\operatorname{Tr}(XY), \operatorname{Tr}(XX)\} + \{\operatorname{Tr}(XY), \operatorname{Tr}(Y)\}\operatorname{Tr}(XX) \\ &= \operatorname{Tr}(Y)(-\operatorname{Tr}(XX) - \operatorname{Tr}(XX)) + \operatorname{Tr}(Y)\operatorname{Tr}(XX) \\ &= -\operatorname{Tr}(XX)\operatorname{Tr}(Y)\end{aligned}$$

C_{42}

What did we know about C_{42}?

- The 32 generators.

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We denote the traces as follows, to ease notation:

$$a_1 = \mathrm{Tr}(X) \quad a_2 = \mathrm{Tr}(Y)$$
$$a_3 = \mathrm{Tr}(A^2) \quad a_4 = \mathrm{Tr}(AB) \quad a_5 = \mathrm{Tr}(B^2)$$
$$a_6 = \mathrm{Tr}(A^3) \quad a_7 = \mathrm{Tr}(A^2B) \quad a_8 = \mathrm{Tr}(AB^2) \quad a_9 = \mathrm{Tr}(B^3)$$
$$\vdots$$
$$a_{32} = Tr([A,B]^3(A^2B^2 - AB^2A - BA^2B + B^2A^2))$$

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For instance:

$$\{\operatorname{Tr}(B^2), \operatorname{Tr}(A^4 B)\}$$

can multiplied directly, or first by applying the Cayley-Hamilton theorem on the second variable,

$$\{\operatorname{Tr}(B^2), \frac{1}{2} \operatorname{Tr}(A^2) \operatorname{Tr}(A^2 B) + \frac{1}{3} \operatorname{Tr}(A^3) \operatorname{Tr}(AB)\}$$

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Since the Poisson bracket has degree $r + s - 2$, in this case, the result will be of degree 5 as well, yielding some identity between traces of degree 5.

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For instance, with the example of before

$$\mathrm{Tr}(A^3B^2) = \frac{1}{12}(a_5a_6 + 6a_4a_7 + 3a_3a_8 - 6a_{16}),$$
$$\mathrm{Tr}(A^2BAB) = \frac{1}{12}(a_5a_6 + 6a_4a_7 + 3a_3a_8 + 6a_{16}).$$

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And this is where the magic happens. If I multiply any relation by $a_5 = \mathrm{Tr}(Y^2)$, since the Poisson bracket has degree $r + s - 2$, I will get a new relation of the same degree.
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On the other hand, if we multiply any relation by $a_6 = \mathrm{Tr}(X^3)$, a relation of degree $+1$ will appear! Therefore, we can now get relations of higher order for free.

Outcome

With these base ideas in mind, we created an algorithm to compute relations of higher order. Since we knew the Hilbert series, we knew when to stop looking in a certain degree and jump to the next one.

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With these base ideas in mind, we created an algorithm to compute relations of higher order. Since we knew the Hilbert series, we knew when to stop looking in a certain degree and jump to the next one.

From degree 12 to 19 we got exactly 104 relations. However, in degree 20 we got just one more, and in degree 21 we did not get anything new. We could not do anything in degree 22 because the computations were too heavy. Therefore, we hoped that we already had them all and tried to prove it.

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Since we knew the Hilbert series, it was just a matter of finding a Gröbner basis, but it was too heavy (we have 30 variables and the file of relations is $2.33Mb$)

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Our relations were inside the ideal we were searching. Since they generate an ideal of the same degrees than the one we were searching, we knew we were done.

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We got 105 relations in total, from degrees 12 to 20.
However, we only needed to find 8 base relations (of degrees 12–16).
From those 8, by using the Poisson bracket with a_5 and a_6 we managed to get the rest.

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We also managed to obtain the Hironaka decomposition, which was an easy task after having all the relations.

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In fact, C_{42} is isomorphic to the GIT quotient $\mathcal{M}_n^2 /\!\!/ \mathrm{GL}_n$.

Let

$$\mathrm{CM}_n := \{(X, Y) \in \mathcal{M}_n^2 \mid \mathrm{rank}([X, Y] + I_n) = 1\}$$

Then the Calogero-Moser space is the GIT quotient $\mathrm{CM}_n /\!\!/ \mathrm{GL}_n$.

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It was also an open question of Nakajima to find loops and elliptic curves there.

References

X. García Martínez, Z. Normatov, R. Turdibaev,
The ring of invariants of pairs of 3×3 matrices,
Journal of Algebra 603(1), 201–212, (2022).

F. Eshmatov, X. García Martínez, Z. Normatov, R. Turdibaev,
On the coordinate rings of Calogero-Moser spaces and the invariant commuting variety of a pair of matrices,
Results in Mathematics 80, Article 68, (2025).

F. Eshmatov, X. García Martínez and R. Turdibaev,
Noncommutative Poisson structure and invariants of matrices,
Advances in Mathematics 469, 110212, (2025).