Coinductive reasoning for parametrized functors and monads

Coimbra Seminar in Algebra, Logic, and Topology

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Coalgebras to model systems

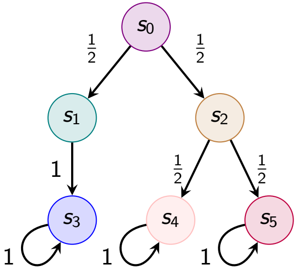
- Systems are modeled as coalgebras for an endodunctor F : **Set** $\rightarrow$ **Set**:

$$(X, \alpha : X \longrightarrow F(X))$$

set of states   behavior of the system

- Examples:
  - Labelled transition systems $X \rightarrow \mathcal{P}(\Sigma \times X)$
  - Streams $X \rightarrow 1 + \Sigma \times X$
  - Deterministic Automata $X \rightarrow 2 \times X^{\Sigma}$
  - Labelled Markov chains $X \rightarrow \mathcal{D}(\Sigma \times X)$

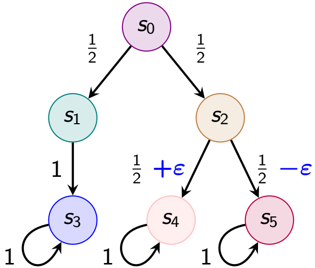
Relations and metrics for systems



behavioral equivalence

$$s_1 \sim s_2$$

$$\sim \subseteq \text{States} \times \text{States}$$



behavioral distance

$$d(s_1, s_2) = \varepsilon$$

$$d : \text{States} \times \text{States} \to [0, 1]$$

General setting

- For endofunctors and monads, we use the theory of lax extensions to reason on behavioral relations

- In this talk: we consider parametrized functors $F : \mathbf{P} \times \mathbf{C} \to \mathbf{C}$ and monads $T : \mathbf{P}^{\mathrm{op}} \times \mathbf{P} \times \mathbf{C} \to \mathbf{C}$ to refine the notions of behavioral equivalence and metrics with parameters

- Objective 1: what is a parametrized lax extension?

- Objective 2: what are the corresponding notions of behavioral relations and metrics?

What is a parametrized lax extension?

Parametrized monads

Atkey, 2006

- For categories $\mathbf{P}$ and $\mathbf{C}$, a parametrized monad consists of:
  - a functor $T : \mathbf{P}^{\mathrm{op}} \times \mathbf{P} \times \mathbf{C} \to \mathbf{C}$
  - a family $\eta_{S,X} : X \to T(S, S, X)$ natural in X and dinatural in S
  - a family $\mu_{S_1,S_2,S_3,X} : T(S_1, S_2, T(S_2, S_3, X)) \to T(S_1, S_3, X)$ natural in X, S_1, S_3 and dinatural in S_2

  satisfying some compatibility axioms.

- They arise from parametrized adjunctions

$$F : \mathbf{P} \times \mathbf{C} \to \mathbf{D} \qquad \dashv \qquad G : \mathbf{P}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$$

$$\mathbf{D}(F(A, X), Y) \;\cong\; \mathbf{C}(X, G(A, Y))$$

Examples

- Parametrized state monad $T : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$:

$$(S_1, S_2, X) \mapsto (S_2 \times X)^{S_1}$$

- Composable continuation monad $T : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$:

$$(R_1, R_2, X) \mapsto (X \Rightarrow R_1) \Rightarrow R_2$$

- Parametrized input-output monad $T : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$:

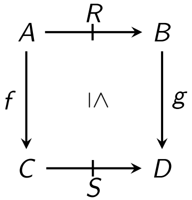
$$(I, O, X) \mapsto \mu Z.(X + Z^I + O \times Z)$$

- The parametrized action monad $\mathbf{P}^{\mathrm{op}} \times \mathbf{P} \times \mathbf{Set} \to \mathbf{Set}$:

$$(A, B, X) \mapsto \mathbf{P}(A, B) \times X$$

Double category of relations $\mathbb{R}\mathbf{el}$

- objects: sets A, B
- vertical maps: functions $f : A \to B$
- horizontal maps: binary relations $R : A \nrightarrow B$ ($R \subseteq A \times B$)
- squares: inclusion of relations

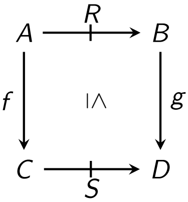


$$\forall a \in A, b \in B, a\, R\, b \Rightarrow f(a)\, S\, g(b)$$

Double category of quantale enriched profunctors $\mathbb{P}\mathbf{rof}_{\mathcal{V}}$

$(\mathcal{V}, \leq, \otimes, 1)$ commutative unital quantale (e.g. $(\{\bot, \top\}, \wedge, \top, \leq)$ and $([0, +\infty], +, 0, \geq)$)

- objects: $\mathcal{V}$-enriched categories $\mathbf{A}, \mathbf{B}$
- vertical maps: $\mathcal{V}$-functors $f : \mathbf{A} \to \mathbf{B}$
- horizontal maps: $\mathcal{V}$-profunctors $R : \mathbf{A} \nrightarrow \mathbf{B}$ i.e. $\mathcal{V}$-functors $\mathbf{A} \otimes \mathbf{B}^{\mathrm{op}} \longrightarrow \mathcal{V}$
- squares: $\mathcal{V}$-transformations



$$R(a, b) \leq S(f(a), g(b))$$

Equipments

- Every function $f : A \to B$ induces two relations

$$f_* = \{(a, fa) \mid a \in A\} : A \nrightarrow B \qquad f^* = \{(fa, a) \mid a \in A\} : B \nrightarrow A$$

- These two relations form an adjunction $f_* \dashv f^*$ in **Rel**.
- We can do the same for enriched profunctors and obtain 2-functors

$$(-)_* : \mathbf{Cat}_{\mathcal{V}} \to \mathbf{Prof}_{\mathcal{V}} \qquad (-)^* : \mathbf{Cat}_{\mathcal{V}}^{\mathrm{op}} \to \mathbf{Prof}_{\mathcal{V}}^{\mathrm{co}}$$

- Note that $(-)^*$ reverses the direction of both 1-morphisms and 2-morphisms

Ordinary strict extensions

- A strict extension for an endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ consists of a functor $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ such that

$$
\begin{array}{ccc}
\mathbf{Rel} & \xrightarrow{\quad \overline{T} \quad} & \mathbf{Rel} \\
{\scriptstyle (-)_*} \uparrow & & \uparrow {\scriptstyle (-)_*} \\
\mathbf{Set} & \xrightarrow[\quad T \quad]{} & \mathbf{Set}
\end{array}
$$

- A strict extension for a monad (T, μ, η) on $\mathbf{Set}$ is a monad $(\overline{T}, \overline{\mu}, \overline{\eta})$ on $\mathbf{Rel}$ with $\overline{T}$ as above and $\overline{\mu}, \overline{\eta}$ have components $(\mu_A)_*$ and $(\eta_A)_*$

- Equivalent formulation: strict double functor or monad $\mathbb{R}\mathbf{el} \to \mathbb{R}\mathbf{el}$

Ordinary lax extensions

A lax extension for a monad (T, μ, η) is a family of functions:

$$\overline{T} : \mathbf{Rel}(X, Y) \longrightarrow \mathbf{Rel}(TX, TY)$$

satisfying the following axioms:

- monotone wrt the inclusion of relations

$$R \leq R' : X \nrightarrow Y \Rightarrow \overline{T}(R) \leq \overline{T}(R')$$

- preserves relation composition laxly

$$\overline{T}(S) \circ \overline{T}(R) \leq \overline{T}(S \circ R)$$

- for all functions $f : X \to Y$

$$(T(f))_* \leq \overline{T}(f_*) \qquad \text{and} \qquad (T(f))^* \leq \overline{T}(f^*)$$

- $(\eta_X)_*$ and $(\mu_X)_*$ are lax natural:

$$(\eta_Y)_* \circ R \leq \overline{T}(R) \circ (\eta_X)_* \qquad (\mu_Y)_* \circ \overline{T}(\overline{T}(R)) \leq \overline{T}(R) \circ (\mu_X)_*$$

Parametrized lax extension

A parametrized lax extension for a functor $T : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ is a family of functions:

$$\mathbf{Rel}(A,B) \times \mathbf{Rel}(C,D) \times \mathbf{Rel}(X,Y) \xrightarrow{\overline{T}} \mathbf{Rel}(T(A,C,X), T(B,D,Y))$$

satisfying the following axioms:

- anti-monotone in the 1rst variable and monotone in the 2nd and 3rd

$$M' \leq M, N \leq N, R \leq R' : X \nrightarrow Y \Rightarrow \overline{T}(M,N,R) \leq \overline{T}(M',N',R')$$

- preserves relation composition laxly

$$\overline{T}(M',N',R') \circ \overline{T}(M,N,R) \leq \overline{T}(M' \circ M, N' \circ N, R' \circ R)$$

- for all functions $f : A \to B$, $g : C \to D$ and $h : X \to Y$

$$(T(f,g,h))_* \leq \overline{T}(f^*, g_*, h_*) \quad \text{and} \quad (T(f,g,h))^* \leq \overline{T}(f_*, g^*, h^*)$$

Double categorical formulation

- Vertical (or tight) opposite of a double category:

$$\begin{array}{ccc} A & \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & B \\ {\scriptstyle f}\big\downarrow & \Downarrow & \big\downarrow{\scriptstyle g} \\ C & \underset{S}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & D \end{array} \quad \text{in } \mathbb{D} \qquad \Longleftrightarrow \qquad \begin{array}{ccc} C & \overset{S}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & D \\ {\scriptstyle f}\big\downarrow & \Downarrow & \big\downarrow{\scriptstyle g} \\ A & \underset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & B \end{array} \quad \text{in } \mathbb{D}^{\mathrm{vop}}$$

- A strict lax extension is a strict double functor

$$\mathbb{R}\mathbf{el}^{\mathrm{vop}} \times \mathbb{R}\mathbf{el} \times \mathbb{R}\mathbf{el} \to \mathbb{R}\mathbf{el}$$

- A parametrized lax extension is a lax double functor

$$\mathbb{R}\mathbf{el}^{\mathrm{vop}} \times \mathbb{R}\mathbf{el} \times \mathbb{R}\mathbf{el} \to \mathbb{R}\mathbf{el}$$

Example: parametrized state monad

The functor $T : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$:

$$(A, C, X) \mapsto (C \times X)^A$$

has a parametrized lax extension $\overline{T}$ which maps a triple of relations

$$(M : A \rightarrow\!\!\!\!\!\!+\;\, B, N : C \rightarrow\!\!\!\!\!\!+\;\, D, R : X \rightarrow\!\!\!\!\!\!+\;\, Y)$$

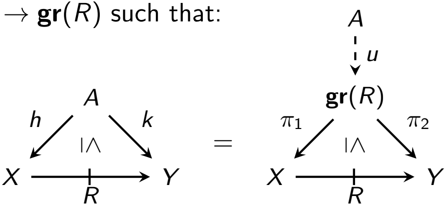
to $\overline{T}(M, N, R) : (C \times X)^A \rightarrow\!\!\!\!\!\!+\;\, (D \times Y)^B$ given by

$$\{(f, g) \mid \forall (a, b) \in M, (f(a), g(b)) \in N \times R\}$$

$$\begin{array}{ccc} A & \xrightarrow[\quad]{M} & B \\ {\scriptstyle f}\downarrow & \leq & \downarrow{\scriptstyle g} \\ C \times X & \xrightarrow[N \times R]{} & D \times Y \end{array}$$

Tabulators

- A relation $R : A \nrightarrow B$ can be represented by its graph $\mathbf{gr}(R) := \{(a,b) \mid (a,b) \in R\} \subseteq A \times B$ and two projections $\pi_1 : \mathbf{gr}(R) \to A$, $\pi_2 : \mathbf{gr}(R) \to B$
- tabulator universal property: for every other span of functions $(h : A \to X, k : A \to Y)$ as on the left, there exists a unique function $u : A \to \mathbf{gr}(R)$ such that:



- tabulators are strong: we can recover the original relation

$$A \xrightarrow{(\pi_1)^*} \mathbf{gr}(R) \xrightarrow{(\pi_2)_*} B \; = R$$

The Barr extension

When do extensions exist and when are they unique?

The Barr extension

When do extensions exist and when are they unique?

The Barr extension is the operator defined by:

$$X \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y \qquad \mapsto \qquad \overline{T}(R) := \; T(X) \xrightarrow{(T\pi_1^R)^*} T(\mathbf{gr}(R)) \xrightarrow{(T\pi_2^R)_*} T(Y)$$

Theorem (Barr, Trnková)

*For a functor T : **Set** $\to$ **Set** , TFAE:*

- *T preserves weak-pullbacks*
- *the Barr extension is a strict extension and it is the unique one*

The lattice of Barr extensions

- For a functor $T : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ and a triple of relations $(M : A \relbar\joinrel\mapsto B, N : C \relbar\joinrel\mapsto D, R : X \relbar\joinrel\mapsto Y)$, define relations

$$\mathcal{L}_i(M, N, R) : T(A, C, X) \relbar\joinrel\mapsto T(B, D, Y)$$

$$\mathcal{L}_1(M, N, R) := (T(\mathrm{id}, \pi_1^N, \pi_1^R))^*; (T(\pi_1^M, \mathrm{id}, \mathrm{id}))_*; (T(\pi_2^M, \mathrm{id}, \mathrm{id}))^*; (T(\mathrm{id}, \pi_2^N, \pi_2^R))_*$$

$$\mathcal{L}_2(M, N, R) := (T(\pi_1^M, \mathrm{id}, \mathrm{id}))_*; T(\pi_2^M, \pi_1^N, \pi_1^R))^*; (T(\mathrm{id}, \pi_2^N, \pi_2^R))_*$$

$$\mathcal{L}_3(M, N, R) := (T(\mathrm{id}, \pi_1^N, \pi_1^R()))^*; (T(\pi_1^M, \pi_2^N, \pi_2^R))_*; (T(\pi_2^M, \mathrm{id}, \mathrm{id}))^*$$

$$\mathcal{L}_4(M, N, R) := (T(\pi_1^M, \mathrm{id}, \mathrm{id}))_*; (T(\mathrm{id}, \pi_1^N, \pi_1^R))^*; (T(\mathrm{id}, \pi_2^N, \pi_2^R))_*; (T(\pi_2^M, \mathrm{id}, \mathrm{id}))^*$$

- We obtain a lattice of Barr extensions:

$$\mathcal{L}_1 \leq \mathcal{L}_2, \quad \mathcal{L}_1 \leq \mathcal{L}_3, \quad \mathcal{L}_2 \leq \mathcal{L}_4, \quad \mathcal{L}_3 \leq \mathcal{L}_4$$

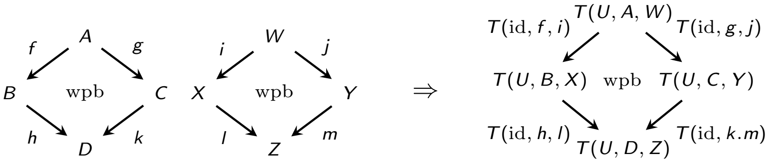
The lattice of Barr extensions

For $i \in \{1,2,3,4\}$, the Barr extension $\mathcal{L}_i$ satisfies:

- it is normal: $\mathrm{id}_{T(A,B,X)} = \mathcal{L}_i(\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_X)$
- it is anti-monotone in the 1rst argument and monotone 2nd and 3rd argument:
  $M' \leq M, N \leq N, R \leq R' : X \nrightarrow Y \Rightarrow \overline{T}(M,N,R) \leq \overline{T}(M',N',R')$
- it preserves composition laxly in the first argument:
  $\mathcal{L}_i(M', \mathrm{id}, \mathrm{id}) \circ \mathcal{L}_i(M, \mathrm{id}, \mathrm{id}) \leq \mathcal{L}_i(M' \circ M, \mathrm{id}, \mathrm{id})$
- it preserves composition op-laxly in the last two arguments:
  $\mathcal{L}_i(\mathrm{id}, N' \circ N, R' \circ R) \leq \mathcal{L}_i(\mathrm{id}, N', R') \circ \mathcal{L}_i(\mathrm{id}, N, R)$
- for all functions $f : A \to B$, $g : C \to D$ and $h : X \to Y$:
  $(T(f,g,h))_* = \mathcal{L}_i(f^*, g_*, h_*)$ and $(T(f,g,h))^* = \mathcal{L}_i(f_*, g^*, h^*)$
- for any lax extension $\mathcal{L}$ of T and relations M, N, R, we have
  $\mathcal{L}_i(M,N,R) \leq \mathcal{L}(M,N,R)$

Towards lax extensions

If T preserves weak pullbacks in the last two arguments



then for each $i \in \{1, 2, 3, 4\}$, $\mathcal{L}_i$ preserves composition laxly in the last two arguments:

$$\mathcal{L}_i(\mathrm{id}, N', R') \circ \mathcal{L}_i(\mathrm{id}, N, R) \leq \mathcal{L}_i(\mathrm{id}, N' \circ N, R' \circ R).$$

Cubical functors

- The two inclusions

$$\mathcal{L}_i(\mathrm{id}, N', R') \circ \mathcal{L}_i(\mathrm{id}, N, R) \leq \mathcal{L}_i(\mathrm{id}, N' \circ N, R' \circ R)$$
$$\mathcal{L}_i(M', \mathrm{id}, \mathrm{id}) \circ \mathcal{L}_i(M, \mathrm{id}, \mathrm{id}) \leq \mathcal{L}_i(M' \circ M, \mathrm{id}, \mathrm{id})$$

  do not imply

$$\mathcal{L}_i(M', N', R') \circ \mathcal{L}_i(M, N, R) \leq \mathcal{L}_i(M' \circ M, N' \circ N, R' \circ R)$$

- For general 2-categories $\mathscr{B}, \mathscr{C}, \mathscr{D}$ and a bifunctor $F : \mathscr{B} \times \mathscr{C} \to \mathscr{D}$, there is an intermediate notion between pseudo and lax functors

$$\text{pseudo} \Longrightarrow \underset{\substack{F(f,\mathrm{id}) \circ F(\mathrm{id}, g) \\ \cong \\ F(\mathrm{id}, g) \circ F(f, \mathrm{id})}}{\text{cubical}} \Longrightarrow \text{lax}$$

Cubicality and strong variable interchange

- T satisfies the strong interchange property if for every $f : A \to B$, $g : C \to D$ and $h : X \to Y$, this square is a weak pullback

$$\begin{array}{ccccc} & & T(B,C,X) & & \\ & \overset{T(f,\mathrm{id},\mathrm{id})}{\swarrow} & & \overset{T(\mathrm{id},g,h)}{\searrow} & \\ T(A,C,X) & & \mathrm{wpb} & & T(B,D,Y) \\ & \underset{T(\mathrm{id},g,h)}{\searrow} & & \underset{T(f,\mathrm{id},\mathrm{id})}{\swarrow} & \\ & & T(A,D,Y) & & \end{array}$$

- Example: if $T(A,C,X) = F(A) \times G(C,X)$ for some $F : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ and $G : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$, then T satisfies this property

Proposition

If T preserves weak pullbacks in its last two arguments and satisfies the strong interchange property, then all the operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ become equal and are parametrized lax extensions.

Cubicality and weak variable interchange

- T satisfies the weak interchange property if for every spans $(g : P \to C, h : P \to D)$ and $(u : Q \to X, v : Q \to Y)$ and split epi $f : A \to B$, the canonical inclusion below is an equality

$$\begin{aligned} & (T(\mathrm{id}, g, u))^*; (T(f, \mathrm{id}, \mathrm{id}))_*; (T(f, \mathrm{id}, \mathrm{id}))^*; (T(\mathrm{id}, h, v))_* \\ \leq\ & (T(f, \mathrm{id}, \mathrm{id}))_*; (T(\mathrm{id}, g, u))^*; (T(\mathrm{id}, h, v))_*; (T(f, \mathrm{id}, \mathrm{id}))^* \end{aligned}$$

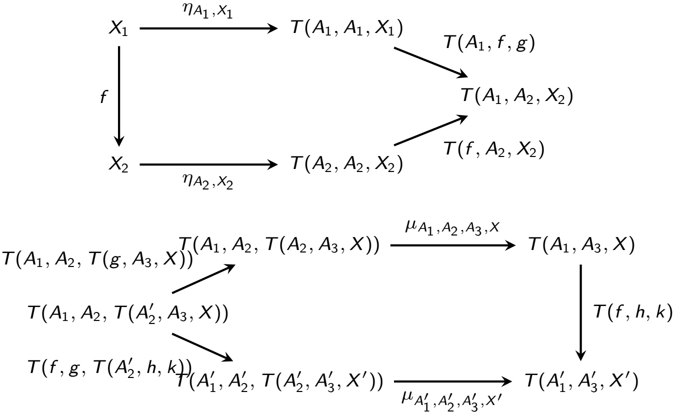
- Example: the state monad $T(A, C, X) = (C \times X)^A$ satisfies the weak interchange property, all the operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ are distinct, and only $\mathcal{L}_4$ is a parametrized lax extension

Proposition

If T preserves weak pullbacks in its last two arguments and and satisfies the weak interchange property, then $\mathcal{L}_4$ is a parametrized lax extension.

Extending parametrized monads

- The unit and multiplication are dinatural in some variables and natural in others:

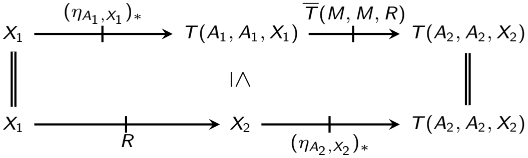


- We have dinaturality wrt "op" in **Set** and dinaturality wrt "co" in **Rel**

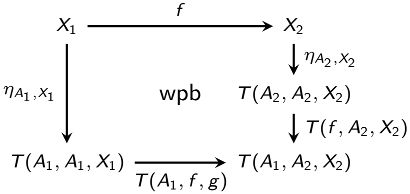
Extending the unit

Fix a parametrized lax extension $\overline{T}$ for T

- The components $(\eta_{A,X})_* : X \nrightarrow T(A,A,X)$ are lax dinatural in A and lax natural in X if for all $M : A_1 \nrightarrow A_2$ and $R : X_1 \nrightarrow X_2$:



- In general, we only obtain the oplax direction. If the following is a weak pullback, we obtain a strict equality



Extending the multiplication

- The family $(\mu_{A_1,A_2,A_3,X})_* : T(A_1, A_2, T(A_2, A_3, X)) \mathrel{\mkern6mu\mapstochar\mkern-6mu\rightarrow} T(A_1, A_3, X)$ is lax dinatural in A_2 and lax natural in A_1, A_3, X if for all $M : A_1 \mathrel{\mkern6mu\mapstochar\mkern-6mu\rightarrow} A_1'$, $N : A_2 \mathrel{\mkern6mu\mapstochar\mkern-6mu\rightarrow} A_2'$, $O : A_3 \mathrel{\mkern6mu\mapstochar\mkern-6mu\rightarrow} A_3'$ and $R : X \mathrel{\mkern6mu\mapstochar\mkern-6mu\rightarrow} X'$:

$$
\begin{array}{ccccc}
T(A_1, A_2, T(A_2, A_3, X)) & \xrightarrow{(\mu_{A_1,A_2,A_3,X})_*} & T(A_1, A_3, X) & \xrightarrow{\overline{T}(M, O, R)} & T(A_1', A_3', X') \\
\| & & \rotatebox{90}{$\geq$} & & \| \\
T(A_1, A_2, T(A_2, A_3, X)) & \xrightarrow[\overline{T}(M, N, \overline{T}(N, O, R))]{} & T(A_1', A_2', T(A_2', A_3', X')) & \xrightarrow[(\mu_{A_1',A_2',A_3',X'})_*]{} & T(A_1', A_3', X')
\end{array}
$$

- If $\overline{T}$ is normal, we only obtain the oplax direction. If the following is a weak pullback, we obtain a strict equality

$$
\begin{array}{ccc}
T(A_1, A_2, T(A_2', A_3, X)) & \xrightarrow{T(A_1, A_2, T(g, A_3, X))} & T(A_1, A_2, T(A_2, A_3, X)) \\
& & \downarrow \mu_{A_1,A_2,A_3,X} \\
T(f, g, T(A_2', h, k)) \downarrow & \text{wpb} & T(A_1, A_3, X) \\
& & \downarrow T(f, h, k) \\
T(A_1', A_2', T(A_2', A_3', X')) & \xrightarrow[\mu_{A_1',A_2',A_3',X'}]{} & T(A_1', A_3', X')
\end{array}
$$

Enriched profunctors

- For a 2-functor $T : \mathbf{Cat}_{\mathcal{V}}^{\mathrm{op}} \times \mathbf{Cat}_{\mathcal{V}} \times \mathbf{Cat}_{\mathcal{V}} \to \mathbf{Cat}_{\mathcal{V}}$, a parametrized lax extension is a lax double functor

$$\mathbb{P}\mathbf{rof}_{\mathcal{V}}^{\mathrm{vop}} \times \mathbb{P}\mathbf{rof}_{\mathcal{V}} \times \mathbb{P}\mathbf{rof}_{\mathcal{V}} \to \mathbb{P}\mathbf{rof}_{\mathcal{V}}$$

- A parametrized strict extension is a strict double functor and in particular, we obtain a 2-functor

$$\overline{T} : \mathbf{Prof}_{\mathcal{V}}^{\mathrm{co}} \times \mathbf{Prof}_{\mathcal{V}} \times \mathbf{Prof}_{\mathcal{V}} \to \mathbf{Prof}_{\mathcal{V}} \text{ and not}$$
$$\overline{T} : \mathbf{Prof}_{\mathcal{V}}^{\mathrm{op}} \times \mathbf{Prof}_{\mathcal{V}} \times \mathbf{Prof}_{\mathcal{V}} \to \mathbf{Prof}_{\mathcal{V}}$$

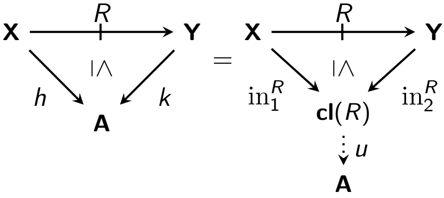
- $\mathbf{Prof}_{\mathcal{V}}$ does not have (strong) tabulators in general and we use cotabulators instead

Cotabulators (collage)

A $\mathcal{V}$-profunctor $R : \mathbf{X} \nrightarrow \mathbf{Y}$ induces a cospan of $\mathcal{V}$-functors $(\mathrm{in}_1^R : \mathbf{X} \to \mathbf{cl}(R), \mathrm{in}_2^R : \mathbf{Y} \to \mathbf{cl}(R))$ where $\mathbf{cl}(R)$ has objects $\mathbf{X}_0 + \mathbf{Y}_0$ and homs:

$$\mathbf{cl}(R)((1,x),(1,x')) := \mathbf{X}(x,x') \quad \mathbf{cl}(R)((2,y),(2,y')) := \mathbf{Y}(y,y')$$
$$\mathbf{cl}(R)((2,y),(1,x)) := R(x,y) \qquad \mathbf{cl}(R)((1,x),(2,y)) := \bot$$

- universal property: for every other cospan $(h : \mathbf{X} \to \mathbf{A}, k : \mathbf{Y} \to \mathbf{A})$ as on the left, there exists a unique $u : \mathbf{cl}(R) \to \mathbf{A}$ such that



- cotabulators are strong: $R = (\mathrm{in}_1^R)_*; (\mathrm{in}_2^R)^*$.

Enriched Barr extension

For a 2-functor $T : \mathbf{Cat}_{\mathcal{V}} \to \mathbf{Cat}_{\mathcal{V}}$, the Barr extension of a $\mathcal{V}$-profunctor $R : \mathbf{X} \looparrowright \mathbf{Y}$ is defined as

$$\overline{T}(R) := \quad T(\mathbf{X}) \xrightarrow{(T\mathrm{in}_1^R)_*} T(\mathbf{cl}(R)) \xrightarrow{(T\mathrm{in}_2^R)^*} T(\mathbf{Y})$$

Theorem (Worrell 2000, Bilkova et al. 2013)

For a 2-functor $T : \mathbf{Cat}_{\mathcal{V}} \to \mathbf{Cat}_{\mathcal{V}}$, if T preserves fully faithful maps, then $\overline{T}$ is a lax extension and if it preserves exact squares then $\overline{T}$ is a strict extension and it is the unique one.

Parametrized Barr extensions

For a 2-functor $T : \mathbf{Cat}_{\mathcal{V}}^{\mathrm{op}} \times \mathbf{Cat}_{\mathcal{V}} \times \mathbf{Cat}_{\mathcal{V}} \to \mathbf{Cat}_{\mathcal{V}}$ and a triple of profunctors ($M : \mathbf{A} \nrightarrow \mathbf{B}$, $N : \mathbf{C} \nrightarrow \mathbf{D}$, $R : \mathbf{X} \nrightarrow \mathbf{Y}$), define $\mathcal{L}_1(M, N, R) : T(A, C, X) \nrightarrow T(B, D, Y)$ as

$$T(\mathrm{in}_1^M, \mathrm{id}, \mathrm{id}))^*; (T(\mathrm{id}, \mathrm{in}_1^N, \mathrm{in}_1^R))_*; (T(\mathrm{id}, \mathrm{in}_2^N, \mathrm{in}_2^R))^*; (T(\mathrm{in}_2^M, \mathrm{id}, \mathrm{id}))_*$$

- We can define similarly $\mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{L}_4$, we obtain a lattice of Barr extensions:

$$\begin{array}{ccc} & \mathcal{L}_1 & \\ \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} \\ \mathcal{L}_2 & & \mathcal{L}_3 \\ \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} \\ & \mathcal{L}_4 & \end{array}$$

- We obtain similar results as before

What is a parametrized behavioral relation?

Behavioral equivalence and final semantics

Two states are behaviorally equivalent if they have the same image in the final coalgebra

A coalgebra $\omega : \Omega \longrightarrow F(\Omega)$ is final if for any other coalgebra $\alpha : X \longrightarrow F(X)$, there is a unique $[\![-]\!] : X \to \Omega$ such that

$$\begin{array}{ccc} X & \xrightarrow{\ \alpha\ } & F(X) \\ {\scriptstyle [\![-]\!]}\downarrow & & \downarrow{\scriptstyle F([\![-]\!])} \\ \Omega & \xrightarrow[\ \omega\ ]{} & F(\Omega) \end{array}$$

Final coalgebra	$\leftrightsquigarrow$	F-behaviors
$[\![x]\!] = [\![y]\!]$	$\leftrightsquigarrow$	x and y are behaviorally equivalent

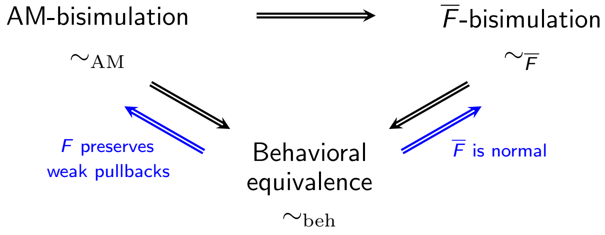
In practice, behavioral equivalence is difficult to establish

Bisimulation proof method

Goal: construct relations that are sound and/or fully abstract for behavioral equivalence

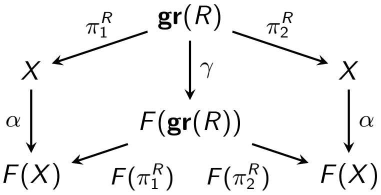
For a coalgebra $\alpha : X \longrightarrow F(X)$, a relation $R : X \rightarrowtail X$

- is sound for behavioral equivalence if $R \subseteq \sim_{\mathrm{beh}}$
- is fully abstract for behavioral equivalence if $R = \sim_{\mathrm{beh}}$



Aczel-Mendler bisimulation

Two states $x \in X$ and $y \in X$ are Aczel-Mendler(AM) bisimilar if there exists a relation $R : X \nrightarrow X$ and a coalgebra $\gamma : \mathbf{gr}(R) \to F(\mathbf{gr}(R))$ such that $(x, y) \in R$ and:



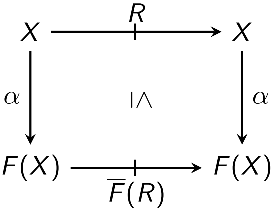
Theorem (Rutten, 2000)

AM-bisimilarity is sound for behavioral equivalence and if F preserves weak-pullbacks, then it is fully abstract.

Bisimulation from lax extensions

- Fix a lax extension $\overline{F}$ for $F : \mathbf{Set} \to \mathbf{Set}$

- For a coalgebra $X \xrightarrow{\alpha} F(X)$, a relation $R : X \relbar\joinrel\mapsto X$ is an $\overline{F}$-bisimulation if



$$\forall x, y \in X, \quad x\,R\,y \qquad \Rightarrow \qquad \alpha(x)\,\overline{F}(R)\,\alpha(y)$$

Induced coinduction principle

- Define $\overline{F}$-bisimilarity relation $\sim_{\overline{F}} : X \nrightarrow X$ as the largest $\overline{F}$-bisimulation

$$\sim_{\overline{F}} := \bigvee \{R : X \nrightarrow X \mid R \text{ is an } \overline{F}\text{-bisimulation}\}$$

Theorem (Marti, Venema 2015)

$\overline{F}$-bisimilarity is sound for behavioral equivalence and if $\overline{F}$ is normal, then $\overline{F}$-bisimilarity is fully abstract.

- Induced coinduction principle:

$$\frac{R : X \nrightarrow Y \text{ bisimulation} \qquad x\, R\, y}{x \sim_{\mathrm{beh}} y}$$

Labelled transition systems

For a set of labels Σ, a labelled transition system (LTS) consists of:

- a set of states X
- a coalgebra $\alpha : X \longrightarrow \mathcal{P}(\Sigma \times X)$

$$x \xrightarrow{a} x' \qquad :\Leftrightarrow \qquad (a, x') \in \alpha(x)$$

When are two states behaviorally equivalent?

A relation $R : X \rightarrow\!\!\!\!\!+ X$ is a bisimulation if for every $x \, R \, y$:

$$x \xrightarrow{a} x' \qquad \Rightarrow \qquad \exists y', y \xrightarrow{a} y' \text{ and } x' \, R \, y'$$
$$y \xrightarrow{a} y' \qquad \Rightarrow \qquad \exists x', x \xrightarrow{a} x' \text{ and } x' \, R \, y'$$

Parametrized bisimulation from lax extensions

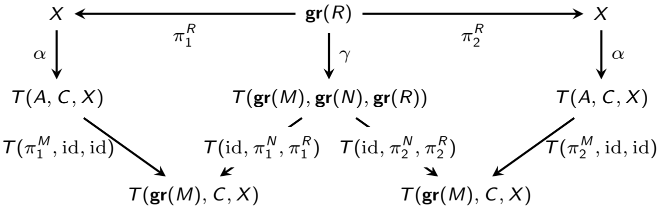
- Fix a parametrized lax extension $\overline{T}$ for $T : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$
- For a coalgebra $\alpha : X \to T(A, B, X)$, fix two relations $M : A \nrightarrow A$ and $N : B \nrightarrow B$ on parameters
- A relation $R : X \nrightarrow X$ is an $\overline{F}$-bisimulation wrt (M, N) if

$$\begin{array}{ccc} X & \overset{R}{\nrightarrow} & X \\ \alpha \downarrow & \leq & \downarrow \alpha \\ T(A,B,X) & \underset{\overline{T}(M,N,R)}{\nrightarrow} & T(A,B,X) \end{array}$$

$$\forall x, y \in X, \quad x\, R\, y \quad \Rightarrow \quad \alpha(x)\, \overline{F}(M, N, R)\, \alpha(y)$$

Parametrized AM-bisimulation

A relation $R : X \not\to Y$ is an AM-bisimulation wrt (M, N) if there exists a coalgebra $\gamma : \mathbf{gr}(R) \to T(\mathbf{gr}(M), \mathbf{gr}(N), \mathbf{gr}(R))$ such that



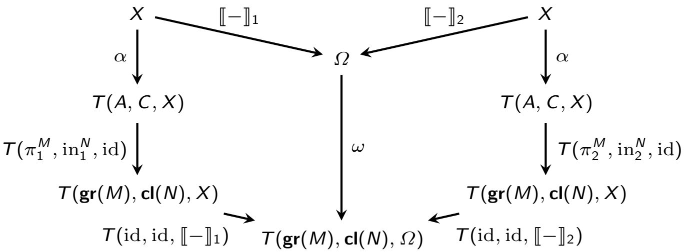
Lemma (soundness)

If R is an AM-bisimulation wrt (M, N), then it is a $\overline{T}$-bisimulation wrt (M, N).

Parametrized behavioral equivalence

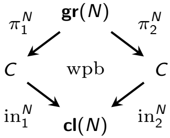
- Assume that the functor $T(\mathbf{gr}(M), \mathbf{cl}(N), -) : \mathbf{Set} \to \mathbf{Set}$ has a final coalgebra $\omega : \Omega \to T(\mathbf{gr}(M), \mathbf{cl}(N), \Omega)$

- Two states $x, y \in X$ are behaviorally equivalent wrt (M,N) if
$$[\![x]\!]_1 = [\![y]\!]_2$$

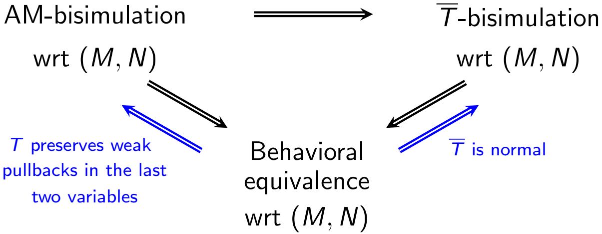


Soundness and completeness

- We restrict to the covariant parameter relation $N : C \to\!\!\!\!\!+ \; C$ to be difunctional



- We obtain



Coinduction for enriched profunctors

Worrell (2000), Goncharov et al. (2022)

Let $F : \mathbf{Prof}_{\mathcal{V}} \to \mathbf{Prof}_{\mathcal{V}}$ be a 2-functor with a lax extension $\overline{F}$:

- $\overline{F}$-bisimilarity is the largest profunctor $R : X \nrightarrow X$ such that

$$\begin{array}{ccc} X & \overset{R}{\nrightarrow} & X \\ {\scriptstyle\alpha}\downarrow & \leq & \downarrow{\scriptstyle\alpha} \\ F(X) & \underset{\overline{F}(R)}{\nrightarrow} & F(X) \end{array}$$

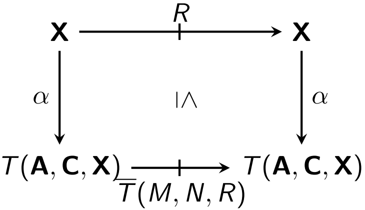
- If F has a final coalgebra $\mathbf{\Omega} \to F(\mathbf{\Omega})$, $\overline{F}$-bisimilarity is the profunctor

$$\begin{aligned} \mathbf{X} \otimes \mathbf{X}^{\mathrm{op}} &\longrightarrow \mathcal{V} \\ (x, y) &\longmapsto \mathbf{\Omega}([\![y]\!], [\![x]\!]) \end{aligned}$$

Parametrized coinduction for enriched profunctors

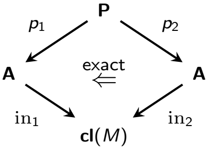
Let $T : \mathbf{Cat}_{\mathcal{V}}^{\mathrm{op}} \times \mathbf{Cat}_{\mathcal{V}} \times \mathbf{Cat}_{\mathcal{V}} \to \mathbf{Cat}_{\mathcal{V}}$ be a 2-functor with a parametrized lax extension $\overline{T}$:

- For a coalgebra $\alpha : X \to T(\mathbf{A}, \mathbf{C}, \mathbf{X})$, define $\overline{T}$-bisimilarity wrt (M, N) as the largest profunctor $R : \mathbf{X} \to\!\!\!\!\!\mapsto \mathbf{X}$ such that



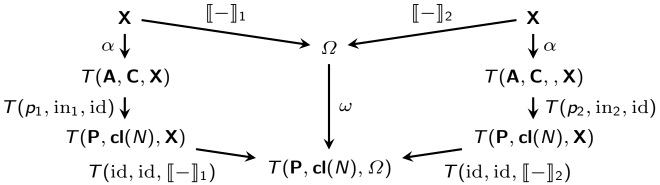
Parametrized coinduction for enriched profunctors

- We restrict to the case where the contravariant parameter profunctor $M : \mathbf{A} \nrightarrow \mathbf{A}$ is represented by an exact square:



- If $T(\mathbf{P}, \mathbf{cl}(N), -)$ has a final coalgebra $\mathbf{\Omega} \to (\mathbf{P}, \mathbf{cl}(N), \mathbf{\Omega})$, $\overline{F}$-bisimilarity wrt (M, N) is the profunctor $\mathbf{X} \otimes \mathbf{X}^{\mathrm{op}} \to \mathcal{V}$

$$(x, y) \longmapsto \mathbf{\Omega}([\![y]\!]_2, [\![x]\!]_1)$$



Performance bisimulation

Corradini, Gorrieri, Roccetti 1995

- Consider a timed LTS $X \longrightarrow \mathcal{P}(\Sigma \times \mathbb{N} \times X)$ with transitions of the form
$$x \xrightarrow{(a,n)} x'$$
- A performance bisimulation is a binary relation $R : X \relbar\joinrel\mapsto X$ such that for all $x \, R \, y$,
$$\begin{array}{lcl} x \xrightarrow{(a,n)} x' & \Rightarrow & \exists m, n \leq m \text{ and } \exists y', y \xrightarrow{(a,m)} y' \text{ and } x' \, R \, y' \\ y \xrightarrow{(a,m)} y' & \Rightarrow & \exists n, n \leq m \text{ and } \exists x', x \xrightarrow{(a,n)} x' \text{ and } x' \, R \, y' \end{array}$$
- Two states are performance equivalent $x \sim y$ if there exists a performance bisimulation R containing (x, y)

Performance bisimulation

- Consider the parametrized functor

$$\begin{aligned} T : \mathbf{Set} \times \mathbf{Set} &\longrightarrow \mathbf{Set} \\ (A, X) &\longmapsto \mathcal{P}(A \times X) \end{aligned}$$

  and its Barr extension $\overline{T} : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$

- A timed LTS is a coalgebra $\alpha : X \longrightarrow T(\Sigma \times \mathbb{N}, X)$.
- Define $M := \{((a, n), (a, m)) \mid a \in \Sigma, m \leq n\} : \Sigma \times \mathbb{N} \nrightarrow \Sigma \times \mathbb{N}$
- A performance bisimulation is exactly a $\overline{T}$-bisimulation wrt M:

$$\begin{array}{ccc} X & \xrightarrow[\quad]{R} & X \\ {\scriptstyle \alpha}\downarrow & \leq & \downarrow{\scriptstyle \alpha} \\ T(\Sigma \times \mathbb{N}, X) & \xrightarrow[\overline{T}(M, R)]{} & T(\Sigma \times \mathbb{N}, X) \end{array}$$

Parametrized computational effects

- We consider a parametrized λ-calculus induced by a parametrized monad T where sequents are of the form

$$\Gamma \vdash_v V : A \qquad \Gamma; S_1 \vdash_c M : A; S_2$$

- The let-rule becomes

$$\frac{\Gamma; S_1 \vdash_c M : A; S_2 \;\; \Gamma, x : A; S_2 \vdash_c N : B; S_3}{\Gamma; S_1 \vdash_c \mathtt{let}\, x = M \,\mathtt{in}\, N : B; S_3}$$

- The operational semantics is an indexed family

$$\{\Downarrow : \Lambda(S_1, S_2, A) \to T(S_1, S_2, \mathcal{V}(A))\}_{A, S_1, S_2}$$

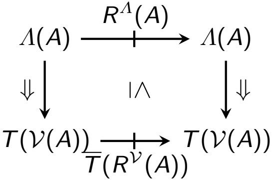
$$\Lambda(S_1, S_2, A) := \{M \mid \emptyset; S_1 \vdash_c M : A; S_2\} \quad \mathcal{V}(A) := \{V \mid \emptyset \vdash_v V : A\}$$

Parametrized contextual equivalence

For an ordinary monad T : **Set** $\to$ **Set** with a lax extension $\overline{T}$, contextual equivalence is a family of relations on values and computations $(R^{\mathcal{V}}, R^{\Lambda})$ coinductively defined as the largest equivalence that

is compatible wrt all term constructors

is adequate wrt the operational semantics



- We quantify over contexts which have access to the whole memory
- In the parametrized setting, we can consider parametrized $\overline{T}$-bisimulation wrt relations on states so that contexts only have access to some partial memory region

Future work

- Parametrized distributive laws
- Connection with predicate liftings
- Connection with bisimulation up-to

Thank you